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I.—ON FRINGES OF INTERFERENCE PRODUCED BY OBLIQUE REFLEXION AT THE SURFACE OF A SMALL MIRROR.

By ANDREW BELL, *Edinburgh.*

THE object of this paper is to determine the circumstances connected with the production of fringes of interference by oblique reflexion at a plane surface, whose projection on a plane perpendicular to the direction of the light, is narrow.

These fringes are produced by light diverging from a minute origin, as a luminous point or line, and are the same as those produced by transmission through an oblique rectangular aperture of the same breadth and inclination to the incident light as the mirror.

The fringes are similar to those produced by narrow perpendicular apertures, with this peculiarity, that the fringes which are nearer to the plane of the inclined aperture or mirror are broader than those which are more distant; the intervals forming an increasing series from one side to the other; the difference in the magnitude of the intervals being very apparent when the inclination or breadth of the mirror, or both, are very small.*

The reflected light is considered to consist of secondary waves emitted from every point of the mirror; and although the waves emitted from the surface at any instant are not then in the same phase, yet the portions of the front of the same primary wave that generate in succession a system of secondary waves, were in the

* Fresnel, in his memoir in the *Memoirs of the Institute* for 1821, alludes in general terms to the cases of oblique reflexion and transmission through oblique apertures; but he does not investigate the case, and he takes no notice of the varying magnitude of the intervals.

same phase when the front of the wave was in some previous position, or when it was emitted from the origin of light; and the paths can be reckoned from that position, or the origin.

If the fringes formed by a single straight edge are called a *single* system, and those formed by a narrow aperture, a *double* system; then the reflected partial waves are under all the necessary conditions for forming a double system of fringes, if the screen is at a sufficient distance from the mirror to allow the partial waves from the whole surface to interfere; and if at the same time the difference of their paths is not too great, reckoning them as formerly stated.

The disturbing effect at any point on a screen produced by the partial waves reflected from an element of the surface, can be estimated in the following manner:—The function $\sin \frac{2\pi}{\lambda}(vt - s)$, which, with a proper coefficient, expresses the state of the particles of ether at any instant along the course of a primary incident wave to some point on the mirror, will also, with its coefficient properly modified, express at the same instant the state of the particles along the course of a secondary wave, generated by the former at that point of the surface. If x is reckoned from the origin, the intensity of vibration at the point on the mirror is inversely as x; and if y is the distance of the point on the screen from that on the mirror, the intensity of vibration at the former point is inversely as y, when that at the latter point is constant: therefore the intensity of vibration at the point on the screen will be inversely as xy, which is nearly constant.

Hence, if Δw denote the breadth of an element of the mirror, and if $s = x + y$; also, if D be the disturbance at the point on the screen, then

$$\mathrm{D} = \mathrm{C} \Sigma \sin \frac{2\pi}{\lambda}(vt - s)\, \Delta w.$$

To determine the intervals on a screen, the section of which by the plane of incidence is a circle, its centre being in the middle of the mirror.

Let RM (fig. 1) be the mirror, O the luminous origin, FQF′ the section of the screen, m the middle of the mirror, mQ the direction in which a wave is reflected which is incident in the direction Om; also, let Ow, wF, be the path of any other wave, and let

RM $= 2b$,	wF $= u$,
$mw = w$,	mF $= u'$,
O$w = r$,	angle OmC $= \beta$,
O$m = r'$,	and FmQ $= \theta$.

In the triangles Omw, Fmw,

$$r^2 = r'^2 + w^2 + 2r'w \cos \beta,$$
$$u^2 = u'^2 + w^2 - 2u'w \cos (\beta - \theta).$$

Hence $$r=r'+w\cos\beta+\frac{\sin^2\beta}{2r'}w^2+\ldots$$

and $$u=u'-w\cos(\beta-\theta)+\frac{\sin^2(\beta-\theta)}{2u'}w^2+\ldots$$

But when a double system of fringes is produced, the greatest value of $w\sin\beta$ must not exceed $\frac{1}{10}$ of an inch if the screen is not more than a few feet distant, suppose not less than 50 inches;* hence, since θ is always small, the third terms in the values of r and u will at most be only a few multiples of λ, and may therefore be omitted. Hence

$$r+u=r'+u'+\{\cos\beta-\cos(\beta-\theta)\}\,w.$$

By the usual method, the intensity of light at any point on the screen is found to be

$$\frac{C^2\sin^2\frac{2\pi}{\lambda}\{\cos\beta-\cos(\beta-\theta)\}\,b}{\frac{4\pi^2}{\lambda^2}\{\cos\beta-\cos(\beta-\theta)\}^2},$$

therefore the points on the screen at which the dark bands appear are determined by the equation

$$\frac{2\pi}{\lambda}\{\cos\beta-\cos(\beta-\theta)\}\,b=n\pi,$$

whence $$\cos(\beta-\theta)=\cos\beta-\frac{n\lambda}{2b}.$$

When $n=0$, $\cos(\beta-\theta)=\cos\beta$; hence $\theta=0$. From this result it would appear that there is a dark band at Q, whereas it is known by observation that the middle point is bright. The ratio of the intensity at this part, and at the bright bands, is determined afterwards.

When n is *negative*, $\cos(\beta-\theta)>\cos\beta$; hence θ is in this case *positive*, and the fringes lie on the side of Q, that is, next to the plane of the mirror towards F.

When n is *positive*, $\cos(\beta-\theta)<\cos\beta$; hence θ is then *negative*, and the fringes lie on the other side of Q towards F′.

By substituting successively for n the values .. 3, 2, 1, 0, −1, −2, −3,... the cosine of $(\beta-\theta)$ has a series of increasing values in equidifferent progression, the common difference being $\frac{\lambda}{2b}$. But if $d.\cos(\beta-\theta)=\frac{\lambda}{2b}$, it is constant, and hence $d\theta\propto\frac{1}{\sin(\beta-\theta)}$;

* Fresnel observed the double system of fringes formed by perpendicular apertures of 1 and even of 1½ centimetres in breadth; but the lens by which he observed them was held at a great distance from the aperture, and at such distances the interference is very imperfect.

and since in the above series $\cos(\beta-\theta)$ is increasing, $(\beta-\theta)$ is diminishing, and hence also $\sin(\beta-\theta)$; therefore the values of $d\theta$, or the intervals on the screen, form an increasing series. This result is confirmed by observation.

When the screen is perpendicular to the plane of the mirror, it is evident that the intervals on it are nearly proportional to

$$d \cdot \tan(\beta-\theta) = -\frac{d \cdot \cos(\beta-\theta)}{\sin(\beta-\theta)\cos^2(\beta-\theta)};$$

that is, the intervals are inversely as

$$\sin(\beta-\theta) \cdot \cos^2(\beta-\theta).$$

Let p and q be given values of the sine and cosine of $(\beta-\theta)$, and let q receive the small increment nk, $\left(k \text{ being } = \frac{\lambda}{2b}\right)$; then, neglecting terms containing k^2, q^2 becomes q^2+2nqk, and

$$p = \sqrt{(1-q^2)} - \frac{nqk}{\sqrt{(1-q^2)}} = p - mk,$$

if $m = \frac{nq}{p}$. Hence for small values of n, as 1, 2, 3, or 4, the increment of pq^2 is, unless in extreme cases, nearly constant; and hence the successive values of

$$\sin(\beta-\theta) \cdot \cos^2(\beta-\theta)$$

for successive values of n, are nearly in equidifferent progression; and their reciprocals, for a few terms, are therefore also nearly in equidifferent progression. And therefore the intervals on the perpendicular screen form a similar progression.

The quantity $\sin(\beta-\theta) \cdot \cos^2(\beta-\theta)$ increases with $(\beta-\theta)$ till it attains its maximum, when $\sin^2(\beta-\theta) = \frac{1}{3}$, and it then diminishes; hence the intervals on the plane screen for greater values of $(\beta-\theta)$ vary in a contrary order, the greater intervals being then farther distant from the plane of the mirror.

The positions of the bright bands are found from the equation

$$\frac{2\pi}{\lambda}\{\cos\beta - \cos(\beta-\theta)\}\, b = \tfrac{1}{2}n\pi,$$

n being an odd number: whence

$$\cos(\beta-\theta) = \cos\beta - \frac{n\lambda}{4b},$$

and the successive increments of $\cos(\beta-\theta)$ are constant for the successive odd values of n, and are $= \frac{\lambda}{2b}$, as for the dark bands.

The expression for the intensity of the illumination at the middle point, for which $\theta=0$, assumes the form $\frac{0}{0}$; its proper value on the

principle of vanishing fractions will be found to be C^2b^2. Also the values of the same quantity for the bright bands in order, are

$$\left(\frac{2Cb}{\pi}\right)^2, \quad \left(\frac{2Cb}{3\pi}\right)^2, \quad \left(\frac{2Cb}{5\pi}\right)^2, \text{ \&c.}$$

$$\text{for } n = \pm 1, \quad \pm 2, \quad \pm 3, \ldots\ldots$$

Therefore the intensities at the middle point, and at the first, second, third, ... bright bands are proportional to

$$\frac{1}{4}, \quad \frac{1}{\pi^2}, \quad \frac{1}{(3\pi)^2}, \quad \frac{1}{(5\pi)^2}, \ldots\ldots$$

the intensities, therefore, diminish rapidly, and soon become insensible.

In a note at p. 404 of his *Memoir on Diffraction*, Fresnel proves that the intensities at two different points produced by the secondary waves, proceeding from the front of the same primary wave, are inversely as the distances of the points from the origin of light; and therefore these intensities are at least proportional to those produced by the primary wave itself when it reaches these points. These results are obtained by supposing the front of the wave to be divided in a particular manner; and by a similar method it can be proved, that the intensities at a point produced by the partial wave from the fronts of two primary waves from the same origin are equal.

Let Wr, W$'r'$, (fig. 2.) be the fronts of two waves proceeding from O, and P any point. Let Wm, W$'m'$, be two arcs of circles, of which P is the centre, and Wn, W$'n'$, two tangents. Also let Pr exceed PW as much as Pr' exceeds PW$'$; let the distance of W from O and P be x and y, W$n = z$, $mr = s$; and let x', y', z', and s' represent the similar lines for the wave W$'r'$; then as the arcs Wr, W$'r'$, are very small,

$$mn = \frac{z^2}{2y} \text{ nearly, } nr = \frac{z^2}{2x}, \text{ hence } s = \frac{(x+y)z^2}{2xy};$$

$$\text{therefore } z^2 = \frac{2xy}{x+y}s, \text{ and hence } z'^2 = \frac{2x'y'}{x'+y'}s,$$

$$\text{for } mr = m'r'; \text{ but } x + y = x' + y':$$

$$\text{therefore } z^2 : z'^2 = xy : x'y'.$$

But the intensities of vibration at W and W$'$ are inversely as x and x'; and the intensities produced at P by partial waves from equal portions of the fronts of the two waves, are inversely as y and y', were the intensities at W and W$'$ equal; also the two circular portions of the fronts, of which z and z' are the radii, are, as shown above, as xy and $x'y'$; and the waves proceeding from these portions are in exactly similar states in regard to accordance or discordance, if the fronts are in the same phase; therefore the

intensities of vibration at P, caused by partial waves from these spaces, being denoted by v and v', it follows that

$$v : v' = \frac{z^2}{xy} : \frac{z'^2}{x'y'} = 1 : 1 ;$$

that is, they are equal.

Pairs of lines, like Pr, Pr', may be drawn from P to the fronts of the waves, such that their differences from the corresponding normals PW, PW′, shall be equal; and the same proportions can be proved to exist in reference to them. And since very small portions of the fronts of the two waves are considered, and xy, $x'y'$, are constant, the corresponding zones into which the fronts would be divided would be proportional; and the waves emitted from the corresponding zones would be in analogous states; and hence the intensities produced by them at P would also be equal. Therefore, the intensities produced at P by partial waves from the fronts of the two primary waves are equal.

In these calculations it is supposed, that the extent of the portion of the front of a wave from which partial waves of sensible intensity proceed, is very small, and that the intensities are uniform, which agrees with observation. But it also implies that the limit of this small surface is determined by a certain difference in the lengths of the paths of the extreme partial waves. It is known by observation, that such a difference alone is a limit to perfect interference, as well as a certain very small inclination. It is also assumed, that at a given distance from the front of a wave, the intensity of vibration of a partial wave is proportional to that in the front of the wave, which is very probable.

The case of transmission through inclined rectangular apertures, is calculated in exactly the same manner as that of oblique reflexion.

If the image of the luminous origin be considered to be the new origin for direct light, and if the inclined mirror be considered to be an oblique aperture, and if partial waves be supposed to proceed from every point in the surface of the mirror, the calculations for this case will be identical with the preceding for oblique reflexion.

The supposition of the emission of secondary waves from the plane of the aperture is equivalent to this hypothesis:—In determining the effect of transmission through narrow oblique apertures, instead of the partial waves being supposed to proceed from the front of a primary wave, they may be conceived to be emitted from the consecutive line of common section of the front with the plane of the narrow aperture, as it advances through the aperture; and the intensity at any point produced by the transmitted light may be calculated by estimating the effect produced by these secondary waves.

I have calculated the breadth of one of the fringes on a circular screen, corresponding to the data of one of my former experiments,

and have found the agreement to be within the limits of what I would consider the unavoidable errors of observation, at least by the means that I have of observing. The difference was about $\frac{1}{200}$ of an inch, or about the tenth part of an interval. It would, I believe, with more perfect apparatus than I have, be possible to determine the intervals more accurately; although there is a limit to the necessary degree of accuracy of apparatus, from the fact, that the eye alone can determine the position of the lines of minimum intensity; and this cannot be done with very great precision.

Most of the experiments that I performed, were only for the purpose of satisfying myself in regard to the general appearance of the fringes. I believe I took accurate measurements only in two or three instances, for the purpose of comparing the observed and calculated intervals

The manner in which I performed the experiments was this:—P (fig. 3.) is a candle or a small mirror reflecting the sun's light. The light passes through O, a vertical aperture in a sheet of pasteboard, and its breadth is about $\frac{1}{40}$ of an inch. M'Q is a screen, a sheet of white pasteboard, placed vertically, and perpendicular to the plane MD of the mirror. When the sun's light is used, the fringes are seen on both sides of Q on the pasteboard; when the light of a candle is used, a lens of about 4 inches focal length, and 3 inches in diameter, is placed at Q; and when the eye is situated behind it, nearly in the principal focus, the fringes are seen very distinctly, and can be measured by applying a pair of finely pointed compasses to the anterior surface of the lens. The measurement by this means will not be so exact as by a micrometer; but the error of observation may not much exceed that arising from estimating by the eye the points of minimum intensity. The breadth of the reflector was measured also by compasses, which were applied to an accurate scale, and by means of a lens the measures could be easily estimated to the $\frac{1}{200}$ part of an inch.

The inclination β can be found with considerable accuracy by taking the lines $mm,'$ mQ, equal, and about 100 inches; and measuring also the line m'Q, and then calculating β.

If RM is the breadth of an inclined aperture, and if the point A in the line OO′, perpendicular to its plane, be found where the extension of its plane meets OO′; then by placing the mirror RM so that the image of O observed from Q may coincide with O′, AO′ being = AO, the mirror will then have the same inclination; and the breadth of the fringes produced by them will be the same. I have at least convinced myself by some experiments that this is the case, though I have not got measurements taken with sufficient precision for publishing them.

One circumstance is apt to mislead if not attended to. If the substance in which the oblique aperture is made is of sensible thickness, and not thinned off to a fine edge, then, the light proceeding in the direction at R, (see fig. 4.) the breadth of the aperture is evidently MR instead of NR; when pasteboard is used, this circumstance produces a very sensible effect with small inclinations.

The mirrors that I used were of plate glass, covered on one side with black wax, and one of speculum metal, $\frac{6}{10}$ of an inch broad.

The formula for the intervals shows that they are independent of the distance of the mirror or aperture from the light, as is the case with narrow obstacles and perpendicular apertures.

II.—NEW SOLUTION OF A CUBIC EQUATION.

By J. Cockle.

There is a soluble form of the perfect cubic equation

$$x^3 + ax^2 + bx + c = 0,$$

to which all cubic equations may be reduced. For if $3ac = b^2$, and the equation be put under the form

$$-x^3 = ax^2 + bx + c,$$

and each side be multiplied by $3ab$,

$$-3abx^3 = 3b.a^2x^2 + 3b^2.ax + b^3, \text{ (since } 3abc = b^3);$$

adding a^3x^3 to each side, and extracting the cube root,

$$x\sqrt[3]{a^3 - 3ab} = ax + b;$$

therefore

$$x = \frac{b}{\sqrt[3]{a^3 - 3ab} - a} \quad \ldots\ldots\ldots(1).$$

In *any* cubic where $3ac$ is not equal to b^2, put $y + z$ for x, and

$$x^3 + ax^2 + bx + c = 0 \text{ becomes}$$

$$y^3 + (3z + a)\,y^2 + (3z^2 + 2az + b)\,y + z^3 + az^2 + bz + c = 0,$$

$$\text{or } \quad y^3 + \mathrm{A}y^2 + \mathrm{B}y + \mathrm{C} = 0,$$

and it is required that $\quad 3\mathrm{AC} = \mathrm{B}^2.$

Putting for A, C, and B, their values in terms of z, &c., and equating; the powers of z above the second vanish, and we have

$$(a^2 - 3b)\,z^2 + (ab - 9c)\,z + (b^2 - 3ac) = 0,$$

$$\text{or}\quad z^2 + \frac{ab - 9c}{a^2 - 3b}z + \frac{b^2 - 3ac}{a^2 - 3b} = 0;$$

therefore z is known, and by (1)

$$y = \frac{\mathrm{B}}{\sqrt[3]{\mathrm{A}^3 - 3\mathrm{AB} - \mathrm{A}}},$$

$$x = y + z = \frac{\mathrm{B}}{\sqrt[3]{\mathrm{A}^3 - 3\mathrm{AB} - \mathrm{A}}} + z.$$

Now $\mathrm{A}^3 - 3\mathrm{AB} = \mathrm{A}(\mathrm{A}^2 - 3\mathrm{B})$

$$= (3z + a)\{(3z + a)^2 - 3(3z^2 + 2az + b)\}$$

$$= (3z + a)(a^2 - 3b);$$

therefore

$$y = \frac{3z^2 + 2az + b}{\sqrt[3]{(a^2 - 3b)(3z + a) - (3z + a)}} = \frac{(3z + a)z + az + b}{\sqrt[3]{(a^2 - 3b)(3z + a) - (3z + a)}}.$$

Hence, if $3z + a = a$,

$$x = \frac{az + az + b}{\sqrt[3]{(a^2 - 3b)\,a - a}} + z \;\ldots\ldots\ldots (2),$$

and (1) may be put under the form

$$x = \frac{b}{\sqrt[3]{(a^2 - 3b)\,a - a}}.$$

As an example, let $x^3 - 4x^2 + 6x - 3 = 0$.

Here $3ac = 36 = 6^2 = b^2$,

therefore $x = \dfrac{b}{\sqrt[3]{a^3 - 3ab - a}} = \dfrac{6}{\sqrt[3]{8 + 4}} = \dfrac{6}{6} = 1.$

Again, $x^3 - 7x^2 + 17x - 14 = 0.$

Here $3ac$ is not equal to b^2; therefore we must apply the second method. The equation for finding z is

$$z^2 + \frac{ab - 9c}{a^2 - 3b}z + \frac{b^2 - 3ac}{a^2 - 3b} = 0,$$

$$\text{or}\quad z^2 + \frac{-119 + 126}{49 - 51}z + \frac{289 - 294}{49 - 51} = 0,$$

$$\text{or}\quad z^2 - \frac{7}{2}z = -\frac{5}{2},\ \text{therefore}\ z = 1\ \text{or}\ \frac{10}{4}.$$

Take $z = 1$, $\quad a = 3z + a = -4$,

and putting $a^2 - 3b = n$,

$$x = \frac{(a + a)z + b}{\sqrt[3]{na - a}} + z = \frac{-11 + 17}{2 + 4} + 1 = \frac{6}{6} + 1 = 2;$$

therefore $x = 2$.

$3\dfrac{b}{a^2}$ might have been used as a multiplier in the commencement instead of $3ab$.

III.—ON THE FORM OF A BENT SPRING.*

In the following papers it is proposed to discover the differential equation to the curve formed by a bent spring, upon the supposition that the particles of the spring are compressible as well as extensible; the present assumed law of tension being only true when the particles can be extended but not compressed.

A spring is supposed to be made up of an indefinite number of indefinitely thin laminæ; and the laminæ composed of an indefinite number of indefinitely short elementary particles, as in fig. 5.

Axioms which are founded upon experiment.

1. If a right line rr', passing through the middle points o, o' of two or more contiguous elementary particles p_1p', qp_1', be perpendicular to their forces pp', p_1p_1', before tension, it shall be so during tension. Did this law not hold the laminæ of the spring would have a sliding motion, and the form of the bent spring could not be made the subject of mathematical investigation.

Cor. All the laminæ of a bent spring have the same normals at the same distances from the ends of the spring; and likewise the centre of curvature of one is the centre of curvature of all the contiguous elementary particles.

2. The particles of springs are extensible and compressible in direction of the laminæ, and it is owing to this state of extension and compression only, that the spring is elastic.

3. If δs be the length of an elementary particle of a lamina before tension, $\delta s'$ after tension; and if T be the tension, and e the elastic modulus,

$$\frac{\delta s' - \delta s}{\delta s} = \frac{\mathrm{T}}{e}.$$

To find the tension of a bent spring at any point, and the force with which it tends to resist bending, let AC (fig. 6.) be the bent spring, QO, Q'O, normals through the middle points of two adjacent laminæ: then these bisect all the contiguous elementary laminæ. Let ll', ll', be the halves of those adjoining elementary laminæ which are neither extended nor compressed, since the result of bending the spring in the ordinary way is to compress those laminæ towards the concave part, and extend those towards the convex.

Then, if lrr' be parallel to QO, $lr''r'''$ parallel to Q'O, these lines define the original boundaries of the elementary particles; consequently if x be the distance of the extremity of a particle from

* From a Correspondent.

ll', that particle has been extended $\frac{x ds}{r}$, if $ds = ll'$, $r =$ rad. of curvature; therefore the force of tension arising from that extension tending to close the extended laminæ, is $\frac{cx\,\delta x \,.\, ds}{r ds}$, if δx be its thickness. Also, its moment tending to turn Qrl' about l into its former position, is $\frac{cx^2\,\delta x}{r}$; therefore $\frac{ca^2}{2r}$ is the force tending to join the spring, if $l'r = a$; and $c\left(\frac{t-a}{2r}\right)^2$ is the force tending to separate the spring if $r'r = t$; therefore $e\left(\frac{a^2-(t-a)^2}{2r}\right)$ is the total force tending to join the spring, or acting upon Qr'' in direction ll'.

Also, $c\left(\frac{a^3+(t-a)^3}{3r}\right)$ is the total force by which Qr' would turn about l' into its former position, were it not that another force keeps it stretched. If the spring be incompressible, this latter force $= \frac{ct^3}{3r}$, which agrees with the ordinary received force.

To find the differential equation to the curve, the thickness of the spring being inconsiderable.

Let P (fig. 7.) be a neutral point, supposed on the spring, R the force applied at the end of the spring; X, Y, the resolved parts of R, in direction of axes of x and y; x, y, the co-ordinates of P: therefore we have

$$X\frac{dx}{ds} + Y\frac{dy}{ds} = -c\left\{\frac{a^2-(t-a)^2}{2r}\right\} \ldots\ldots (1),$$

$$Xy + Yx = c\left\{\frac{a^3+(t-a)^3}{3r}\right\} \ldots\ldots\ldots\ldots (2).$$

In most cases, since a is very small, c must be very large, and the neutral point will lie nearly in the middle of the spring; in this case the ordinary equations hold.

Between (1) and (2) we must eliminate a, and the resulting equation is the equation required. It is to be observed that the resulting equation is the equation to the neutral line. The equation is of the second order, since

$$r = -\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

It is generally more convenient to make the direction of R the axis of y.

In this case we have

$$Y\frac{dy}{ds} = -c\left(\frac{a^2-(t-a)^2}{2r}\right) = \left(\frac{t^2-2at}{2r}\right)\cdot c,$$

$$Yx = c\left(\frac{a^3+(t-a)^3}{3r}\right) = c\left(\frac{t^3-3at^2+3at^2}{3r}\right).$$

Cor. If it be required to discover the law of thickness of a spring, that when bent by a given force it may assume a given form, let $y = f(x)$ be the equation to the curve. We must substitute for $\frac{dy}{ds}$ its value in terms of x, and for r its value in terms of x, whence eliminating a we shall obtain $t_1 = \phi(x)$.

IV.—SINGULAR POINTS IN SURFACES.

By D. F. Gregory, M.A. Fellow of Trinity College.

The nature of those points in surfaces which are analogous to multiple points in curves, has not, so far as I know, been hitherto discussed in any work on Analytical Geometry. Leroy, in his *Analyse Appliquée*, and in his *Géometrie Descriptive*, does little more than indicate their existence; and Monge and Dupin, although they have given much attention to those singular points called *umbilici*, have not spoken at all of the points which are the object of the present article. The subject is, however, one of some interest, from the light which it throws on the form of certain surfaces; and in physical investigations Sir W. Hamilton has shown by his researches on the form of the Wave Surface, that such points are of considerable importance.

In the first place, it is necessary to define precisely what is meant by a *singular* point in the sense in which I use that phrase. If

$$F(x, y, z) = 0 \ldots\ldots\ldots (1),$$

be the equation to a surface, then those points for which all the differential coefficients of F, with respect to x, y, and z, below a certain order, vanish, I call singular points; and the values of x, y, z, which satisfy these conditions, I shall call the *critical* values. We shall have to consider chiefly points for which the differential coefficients, of the first order only, vanish, so that the characteristic of such points is, that the equations

$$\frac{dF}{dx}=0, \quad \frac{dF}{dy}=0, \quad \frac{dF}{dz}=0 \ldots\ldots\ldots (2),$$

are satisfied by simultaneous values of x, y, z, which also satisfy the equation to the surface (1). When the conditions (2) do not hold, we know that the locus of the tangent lines to the surface at a given

point is a plane, which is called the tangent plane. When the conditions (2) do hold, the direction cosines of the tangent take the form $\frac{0}{0}$, and are therefore indeterminate, showing that there may be more than one tangent plane at the point: and we shall see that there are in general an infinite number of such planes, forming by their intersections a tangent *cone*.

Let us now investigate the locus of the tangent lines at a singular point. This we shall do by the same method as that employed in Vol. I. p. 135; and for the convenience of our future operations, let us put

$$\mathrm{U} = \frac{d\mathrm{F}}{dx}, \qquad \mathrm{V} = \frac{d\mathrm{F}}{dy}, \qquad \mathrm{W} = \frac{d\mathrm{F}}{dz},$$

$$u = \frac{d^2\mathrm{F}}{dx^2}, \qquad v = \frac{d^2\mathrm{F}}{dy^2}, \qquad w = \frac{d^2\mathrm{F}}{dz^2},$$

$$u' = \frac{d^2\mathrm{F}}{dy\,dz}, \qquad v' = \frac{d^2\mathrm{F}}{dx\,dz}, \qquad w' = \frac{d^2\mathrm{F}}{dx\,dy}.$$

Let then the equation to a line passing through the point x, y, z, in the surface, be

$$\frac{x' - x}{l} = \frac{y' - y}{m} = \frac{z' - z}{n} = r \ldots\ldots\ldots (a),$$

x', y', z', being the current co-ordinates of the line. This line will generally be cut by the surface in one or more other points. Let the co-ordinates of the nearest of these be x_1, y_1, z_1, then from equation (a) we have

$$x_1 = x + lr, \quad y_1 = y + mr, \quad z_1 = z + nr.$$

Substituting these values in the equation to the surface, it becomes

$$\mathrm{F}\{x + lr, \quad y + mr, \quad z + nr\} = 0.$$

Expanding this, considering lr, mr, nr, as the increments of x, y, z, we have

$$0 = \mathrm{F}(x, y, z,) + r\,(l\mathrm{U} + m\mathrm{V} + n\mathrm{W}) +$$

$$+ \frac{r^2}{1.2}\{l^2u + m^2v + n^2w + 2(mnu' + lnv' + lmw')\} + \frac{r^3}{1.2.3}\{\quad\}.$$

But from the equation to the surface

$$\mathrm{F}(x, y, z,) = 0;$$

and when the point is a singular point,

$$\mathrm{U} = 0, \quad \mathrm{V} = 0, \quad \mathrm{W} = 0;$$

therefore the equation is reduced to

$$0 = \frac{1}{1.2}\{l^2(u) + m^2(v) + n^2(w) + 2(mn(u') - ln(v') + lm(w')\}$$

$$+ \frac{r}{1.2.3}\{\quad\}.$$

where the differentials are inclosed in parentheses to indicate the

values they receive when the critical values of x, y, z, are substituted in them. When the line becomes a tangent, the points x, y, z, x_1, y_1, z_1, ultimately coincide, and r becomes indefinitely small; in which case the preceding equation becomes ultimately

$$l^2(u) + m^2(v) + u^2(w) + 2\{mn(u') + ln(v') + lm(w')\} = 0;$$

whence eliminating l, m, n, by means of equation (a), we find

$$(u)(x'-x)^2 + (v)(y'-y)^2 + (w)(z'-z)^2 + 2\{(w')(y'-y)(z'-z) + (v')(x'-x)(z'-z) + (w')(y'-y)(x'-x)\} = 0 \ldots\ldots (3),$$

as the equation to the locus of the tangent lines. This is evidently in general a cone of the second degree, though with certain values of the coefficients, it may represent two planes or a straight line. On transferring the origin to the singular point, the equation (3) may be put under the form

$$(u)x^2 + (v)y^2 + (w)z^2 + 2\{(u')yz + (v')xz + (w')xy\} = 0 \ldots\ldots (4).$$

If we suppose that all the second differential coefficients also vanish for the critical values of the variables, we should easily find that the locus of the tangent lines at a singular point is a cone of the third order, and so on in succession to any order. The forms of the equations are easily found by taking the corresponding differential of $F(x, y, z)$, and substituting in it x, y, z, for dx, dy, dz. It is unnecessary to write down these expressions, which are rather long, as I shall scarcely have occasion to use them. Before proceeding to give some applications of the formula which has just been investigated, I shall make one or two remarks on the difference, with respect to singular points, between plane curves and surfaces. In the first place it will be readily seen that the three equations (2) may be satisfied by some relation between the variables, without assigning any particular value to each. If we suppose the relation to be given by the equation

$$\phi(x, y, z,) = 0,$$

the intersection of the surface (1), with that represented by the last equation, will give a line which is a *locus* of singular points: that is to say, every point in the line is a singular point. Such, for instance, is the edge of regression of a developable surface, or the rectilinear directrix of a conoid. There is nothing analogous to this in plane curves, since the one variable is always determined in terms of the other by the equation to the curve, so that there is nothing left indeterminate. In the second place, I observed before that the equation (4) may represent a cone of the second order, or two planes, or a straight line, which last may indeed be considered as a cone, the vertical angle of which is zero. In plane curves, the corresponding equation could represent only two straight lines, or two points, since a homogenous function of two variables can always be decomposed into possible or impossible factors of the first degree. We shall now proceed to some examples in illustration of the general theory given above.

1. Let the equation to the surface be

$$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 - c^2z^2.$$

This is the locus of the intersection of the tangent planes to the hyperboloid of one sheet, with perpendiculars on them from the centre. Here

$$U = 2x(2r^2 - a^2), \quad V = 2y(2r^2 - b^2), \quad W = 2z(2r^2 + c^2),$$
$$u = 2(2r^2 - a^2) + 8x^2, \quad v = 2(2r^2 - b^2) + 8y^2, \quad w = 2(2r^2 + c^2) + 8z^2,$$
$$u' = 8yz, \quad v' = 8xz \quad w' = 8xy.$$

At the origin, or when $x = 0$, $y = 0$, $z = 0$, U, V, W, all vanish; that point is therefore a singular point. Substituting the critical values in the expression for the second differentials, we have

$$(u) = -2a^2, \quad (v) = -2b^2, \quad (w) = 2c^2,$$
$$(u') = 0, \quad (v') = 0, \quad (w') = 0.$$

The equation to the tangent cone at the singular point is, therefore,

$$a^2x^2 + b^2y^2 - c^2z^2 = 0.$$

This represents an elliptical cone, the axis of z being perpendicular to the directrix.

2. The equation to Fresnel's Wave Surface is

$$(x^2 + y^2 + z^2)(a^2x^2 + b^2y^2 + c^2z^2) - a^2(b^2 + c^2)x^2 - b^2(a^2 + c^2)y^2 - c^2(a^2 + b^2)z^2 + a^2b^2c^2 = 0.$$

In this case,

$$U = 2x\{a^2(r^2 - b^2 - c^2) + a^2x^2 + b^2y^2 + c^2z^2\},$$
$$V = 2y\{b^2(r^2 - a^2 - c^2) + a^2x^2 + b^2y^2 + c^2z^2\},$$
$$W = 2z\{c^2(r^2 - a^2 - b^2) + a^2x^2 + b^2y^2 + c^2z^2\}.$$

Now these expressions vanish when

$$y = 0, \quad r^2 = b^2,$$

involving as values of x and z

$$x = \pm c\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad z = \pm a\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}.$$

There are therefore four singular points corresponding to the four different ways in which the double signs of x and z may be combined. We find also

$$(u) = 8a^2c^2\frac{a^2 - b^2}{a^2 - c^2}, \quad (v) = -2(a^2 - b^2)(b^2 - c^2), \quad (w) = 8a^2c^2\frac{b^2 - c^2}{a^2 - c^2},$$

$$(u') = 0, \quad (v') = 4ac\sqrt{(a^2 - b^2)(b^2 - c^2)}\,\frac{a^2 + c^2}{a^2 - c^2}, \quad (w') = 0.$$

Substituting these values in equation (4), and putting the result in the simplest form, we find

$$\frac{x^2}{b^2 - c^2} - \frac{a^2 - c^2}{4a^2c^2}y^2 + \frac{z^2}{a^2 - b^2} + \frac{a^2 + c^2}{\sqrt{(a^2 - b^2)(b^2 - c^2)}}\frac{xz}{ac} = 0,$$

as the equation to the tangent cone at each of the singular points. It is supposed throughout that a, b, c, are in order of magnitude.

3. Let the surface be the locus of the intersection of tangent planes to an elliptical paraboloid, with the perpendiculars on them from the vertex, the equation to it being

$$z(x^2+y^2+z^2) + ax^2 + by^2 = 0.$$

Here $U = 2x(z+a)$, $V = 2y(z+b)$, $W = x^2+y^2+3z^2$.

These expressions vanish when $x=0$, $y=0$, $z=0$; which values also satisfy the given equation: the origin is therefore a singular point. We find also

$$(u) = 2a, (v) = 2b, (w) = 0, (u') = 0, (v') = 0, (w') = 0.$$

The equation to the locus of the tangent lines is

$$ax^2 + by^2 = 0.$$

As a and b are both positive, this can be satisfied only by

$$x = 0, \; y = 0;$$

and it therefore represents the axis of z. It will be seen from the equation that z can never be positive since that would render x and y impossible; the point in question is therefore a cusp. The surface surrounds the negative axis of z which it touches at the origin, so that the form of the surface resembles the shape of the flower of the convolvulus. A correct idea of the surface may be formed by supposing a cissoid to revolve round its axis;—the surface generated is the same as that of the surface here discussed when $a = b$. If a and b be of contrary signs, in which case the surface is formed from the hyperbolic paraboloid, the equation to the locus of the tangent lines is

$$ax^2 - by^2 = 0,$$

which represents two planes perpendicular to the plane of x, y.

4. The equation to the Cono-cuneus of Wallis is

$$a^2y^2 - x^2(c^2 - z^2) = 0.$$

Here

$$U = -2x(c^2 - z^2), \quad V = 2a^2y, \quad W = 2x^2z.$$

These all vanish when $x = 0$, $y = 0$, independently of any value of z; so that the axis of z is a locus of singular points or a singular line. We find also

$$(u) = -2(c^2 - z^2), \quad (v) = 2a^2, \quad (w) = 0.$$
$$(u') = 0, \qquad (v') = 0, \quad (w') = 0,$$

so that the equation to the locus of the tangent lines is

$$a^2y'^2 - (c^2 - z^2)x'^2 = 0,$$

the current coordinates being accentuated to distinguish them from z, the undetermined coordinate of the point under consideration. So long as $z < c$ this equation represents two planes perpendicular to the plane of xy. z cannot be greater than c, as y would then become impossible.

5. Let the surface to the *héliçoide dévelopable*, the equation to which is

$$x \sin\left(\frac{2\pi}{h}z - \frac{\sqrt{(x^2+y^2-a^2)}}{a}\right) + y \cos\left(\frac{2\pi}{h}z - \frac{\sqrt{(x^2+y^2-a^2)}}{a}\right) = a.$$

If we assume

$$\frac{2\pi}{h}z - \frac{\sqrt{(x^2+y^2-a^2)}}{a} = \theta,$$

we find

$$\mathrm{U} = \sin\theta - \frac{x(x\cos\theta - y\sin\theta)}{a\sqrt{(x^2+y^2-a^2)}},$$

$$\mathrm{V} = \cos\theta - \frac{y(x\cos\theta - y\sin\theta)}{a\sqrt{(x^2+y^2+a^2)}},$$

$$\mathrm{W} = \frac{2\pi}{h}(x\cos\theta - y\sin\theta).$$

From the equation to the surface it is easily seen that

$$x\cos\theta - y\sin\theta = \sqrt{(x^2+y^2-a^2)}.$$

Hence if we assume

$$x = a\sin\frac{2\pi z}{h}, \qquad y = a\cos\frac{2\pi z}{h},$$

the three preceding expressions will vanish, and therefore the line determined by these equations, and the equation to the surface, is a locus of singular points. This line is the intersection of the surface by the cylinder

$$x^2 + y^2 = a^2,$$

and is evidently the generating helix. Since in the equation to the surface x^2+y^2 can never be less than a^2, it appears that no part of the surface lies within the helix, which is therefore truly an edge of regression.

On proceeding to the second differential coefficients, and substituting in them the critical values of x and y, we find, retaining only the terms which become infinite from involving $\sqrt{(x^2+y^2-a^2)}$ in the denominator,

$$(u) = -2\sin\frac{2\pi z}{h}\cos\frac{2\pi z}{h}, \quad (v) = 2\sin\frac{2\pi z}{h}\cos\frac{2\pi z}{h}, \quad (w) = 0,$$

$$(u') = \frac{2\pi}{h}\cos\frac{2\pi z}{h}, \quad (v') = \frac{2\pi}{h}\sin\frac{2\pi z}{h},$$

$$(w') = \sin^2\frac{2\pi z}{h} - \cos^2\frac{2\pi z}{h},$$

so that the equation to the locus of the tangent lines is

$$(y'^2 - x'^2)\sin nz\cos nz - x'y'(\cos^2 nz - \sin^2 nz) + nz'(x'\sin nz + y'\cos nz) = 0,$$

where $n = \frac{2\pi}{h}$, and the accentuated letters are the current co-ordinates of the tangents, and the unaccentuated z is the undetermined co-ordinate of the point of contact.

This equation may be decomposed into two factors,

$$x' \sin nz + y' \cos nz = 0,$$

and $$y' \sin nz - x' \cos nz + nz' = 0,$$

which are the equations to two planes.

6. I shall take a single example where it is necessary to proceed to an order of differentiation higher than the second.

Let the equation to the surface be

$$(x^2 + y^2 + z^2)^3 = a^2 (y^2z^2 + x^2z^2 + x^2y^2).$$

It is easy to see that the origin is a singular point; and on making $x = 0$, $y = 0$, $z = 0$, all the differential coefficients of the second and third orders vanish, while of the fourth all vanish except three, which are

$$\frac{d^4F}{dx^2dy^2} = \frac{d^4F}{dx^2dz^2} = \frac{d^4F}{dy^2dz^2} = -4a^2.$$

The equation to the locus of the tangent lines is therefore

$$y^2z^2 + x^2z^2 + x^2y^2 = 0.$$

This is satisfied in three ways only, either by

$$x = 0, \quad y = 0,$$

or by $$x = 0, \quad z = 0,$$

or by $$y = 0, \quad z = 0.$$

It therefore represents the three axes, and shows that the surface at the origin touches these lines, forming round each a figure resembling that produced by the revolution of a circular arc round a tangent.

It is needless to multiply examples, more especially as there are few surfaces of a high order which are sufficiently important to require a minute discussion.

V.—ON FOURIER'S EXPANSIONS OF FUNCTIONS IN TRIGONOMETRICAL SERIES.*

The object of the following paper is to endeavour to shew in what cases a function, arbitrary between certain limits, can be developed in a series of cosines, and in what cases in a series of sines. Though the whole subject has been treated very fully by

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* From a Correspondent.

Fourier, yet, as he nowhere directly demonstrates, that a function can be developed in a series of sines or cosines separately, and as, from the want of this direct demonstration, many of his formulæ have been believed to be erroneous, the following paper may be interesting to some readers.

It has been rigorously demonstrated, first, so far as I know, by Fourier and afterwards by Poisson, that

$$\pi fx = \tfrac{1}{2}\int_0^{2\pi} fa\,da + \cos x\int_0^{2\pi} \cos a\,fa\,da + \ldots\ldots\ldots + \sin x\int_0^{2\pi} \sin a\,fa\,da + \ldots\ldots\ldots,$$

where fx is completely arbitrary, between the limits $x = 0$ and $x = 2\pi$. Now, putting the first part of this series equal to $\pi\phi x$, and the second to $\pi\psi x$, we have

$$fx = \phi x + \psi x \ldots\ldots\ldots (a).$$

But, since $\sin nx = -\sin n(2\pi - x)$, and $\cos nx = \cos n(2\pi - x)$, it follows that $\psi x = -\psi(2\pi - x)$, and $\phi x = \phi(2\pi - x)$, and hence

$$f(2\pi - x) = \phi x - \psi x \ldots\ldots\ldots\ldots (b)$$

Hence, by this equation, and (a), we have

$$\tfrac{1}{2}\{fx + f(2\pi - x)\} = \phi x \ldots\ldots\ldots (c)$$

and

$$\tfrac{1}{2}\{fx - f(2\pi - x)\} = \psi x \ldots\ldots\ldots (d).$$

Now, when x is between 0 and π, fx and $f(2\pi - x)$ are perfectly arbitrary, and independent of one another; and therefore it follows that ϕx and ψx are likewise perfectly arbitrary between the same limits. Now

$$\pi\phi x = \tfrac{1}{2}\int_0^{2\pi} fa\,da + \cos x\int_0^{2\pi} \cos a\,fa\,da + \cos 2x\int_0^{2\pi} \cos 2a\,fa\,da + \&c.$$

$$= \tfrac{1}{2}\int_0^{\pi} fa\,da + \cos x\int_0^{\pi} \cos a\,fa\,da + \cos 2x\int_0^{\pi} \cos 2a\,fa\,da + \&c.$$

$$+ \tfrac{1}{2}\int_0^{\pi} f(2\pi - a)da + \cos x\int_0^{\pi} \cos a\,f(2\pi - a)\,da + \cos 2x\int_0^{\pi} \cos 2a\,f(2\pi - a)\,da + \&c.$$

$$= \tfrac{1}{2}\int_0^{\pi} \{fa + f(2\pi - a)\}da + \cos x\int_0^{\pi} \cos a\{fa + f(2\pi - a)\}\,da + \cos 2x\int_0^{\pi} \cos 2a\{fa + f(2\pi - a)\}da + \&c.$$

or by (c),

$$\frac{\pi}{2}\phi x = \tfrac{1}{2}\int_0^{\pi} \phi a\, da + \cos x \int_0^{\pi} \cos a\, \phi a\, da + \cos 2x \int_0^{\pi} \cos 2a\, \phi a\, da + \&c.$$

In a similar manner it may be shewn, that

$$\frac{\pi}{2}\psi x = \sin x \int_0^{\pi} \sin a\, \psi a\, da + \sin 2x \int_0^{\pi} \sin 2a\, \psi a\, da + \&c.$$

Hence we see, that any function whatever of x, may be represented between the limits 0 and π, by a series either of cosines or sines of multiples of x. If it be represented by a series of cosines, then the value of the function corresponding to any value of x, will be found from its values between the limits $x = 0$ and $x = \pi$, by the condition,

$$fx = f(2\pi - x):$$

while if it be represented by a series of sines, the condition will be

$$fx = -f(2\pi - x).$$

The same results might have been obtained by taking, in the original equation, fx subject to either of these conditions. If it be subject to the former, $\int_0^{2\pi} fa \sin na\, da$ will vanish, and fx will be expanded in a series of cosines. If it be subject to the second condition $\int_0^{2\pi} fa \cos na\, da$ will vanish, and fx will be expanded in a series of sines. The two series thus obtained will be equal to one another, when x is between 0 and π; but when x is between π and 2π, the value of one will be the negative of the value of the other.

Fourier gives many expansions of functions in series of sines or cosines alone, obtained from the formulæ given above; which, however he demonstrates merely by assuming fx to be developed in such a series, and then determining the coefficients. Now with regard to these series, Mr. Kelland, in his excellent Treatise on Heat, remarks, that they are "nearly all erroneous." This remark has probably arisen from finding that they differ from the developments obtained from the general formula for functions which follow the given assumption through their whole periods from 0 to 2π, instead of following it merely between the limits 0 and π, as is the case if sines or cosines alone be used.

Thus, Mr. Kelland gives the following expansion of a function which is equal to c, when x is between 0 and a, and to zero, when x is between a and 2π:

$$\pi\frac{\phi x}{c} = \tfrac{1}{2}a + \sin a \cos x + \tfrac{1}{2}\sin 2a \cos 2x + \ldots\ldots + \operatorname{vers} a \sin x + \tfrac{1}{2}\operatorname{vers} 2a \sin 2x + \ldots\ldots;$$

an expression differing, as he remarks, from Fourier's, "which em-

braces only the second line of this." Now, as long as x lies between 0 and π, the two series of which this expression is composed, are equal to one another, if we suppose, for the present, $a < \pi$, and their sum is the required function between these limits. When, however, x is between π and 2π, the value of the one series is the negative of the value of the other, as is readily seen from what has been said above. Hence, when x is between π and 2π the value of the expression is nothing. Now Fourier proposes to find an expansion of a function which is equal to c, when x is between 0 and a, and to zero when x is between a and π, without any supposition regarding the values of the function, when x is between π and 2π. Now it is clear, that the double of either of the series in the expansion given by Mr. Kelland, will be sufficient for this purpose, and Fourier, "Théorie de la chaleur," page 244, chooses to make use of the second; that is, he develops the required function in a series of sines of multiples of x.

The series given by Fourier may be verified in the following manner:

$$\begin{aligned} \text{Let } u &= \text{vers } a \sin x + \tfrac{1}{2} \text{ vers } 2a \sin 2x + \&\text{c.} \\ &= \sin x + \tfrac{1}{2} \sin 2x + \&\text{c.} \\ &\quad - \tfrac{1}{2}\{\sin(a+x) + \tfrac{1}{2}\sin 2(a+x) + \&\text{c.}\} \\ &\quad + \tfrac{1}{2}\{\sin(a-x) + \tfrac{1}{2}\sin 2(a-x) + \&\text{c.}\} \end{aligned}$$

Now it is well known, and it is demonstrated by Mr. Kelland, p. 59, that

$$\sin\theta + \tfrac{1}{2}\sin 2\theta + \&\text{c.} = \tfrac{1}{2}(\pi - \theta),$$

which obviously holds for any value of θ between 0 and 2π, and for no others. Hence, x and a being, of course, each less than π, we have, when $x < a$, $\qquad u = \tfrac{1}{2}\pi.$

When $x > a$, the last series may be put under the form

$$-\tfrac{1}{2}\{\sin(x-a) + \tfrac{1}{2}\sin 2(x-a) + \&\text{c.}\},$$

and consequently we have

$$u = 0.$$

The values of u are thus found for all values of a and x between 0 and π; and, from them, those for any values of a and x are readily found.

In exactly a similar manner it may be shown, that the series given by Mr. Kelland, p. 64, is really the expansion of a very different function from the one of which the series given by Fourier, p. 246, is the expansion. Thus, the trapezium represented by Mr. Kelland's series occupies the whole space along the axis of x, from 0 to 2π; while, in that space, two trapeziums are represented by Fourier's, one extending from 0 to π, and the other from π to 2π, and inverted, with regard to the axes both of x and y. It may also be remarked, that the form of the trapezium represented by Mr. Kelland's series is much more general than that of the trapezium represented by Fourier's.

Fourier's series may be verified in the following manner:

$$\text{Let } u = \frac{4}{\pi}(\sin a \sin x + \frac{1}{3^2}\sin 3a \sin 3x + \ldots).$$

$$\text{Then } \frac{du}{dx} = \frac{4}{\pi}(\sin a \cos x + \frac{1}{3}\sin 3a \cos 3x + \ldots).$$

$$= \frac{2}{\pi}\left\{\begin{array}{l}\sin(a+x) + \frac{1}{3}\sin 3(a+x) + \ldots \\ + \sin(a-x) + \frac{1}{3}\sin 3(a-x) + \ldots\end{array}\right\}$$

Now, if a and x be each less than $\frac{1}{2}\pi$, we have (Fourier, p. 237, or Kelland, p. 65,)

$$\frac{du}{dx} = 1, \text{ when } x < a,$$

$$\text{and } \frac{du}{dx} = 0, \text{ when } x > a.$$

Hence we have, obviously,

$$u = x, \text{ from } x = 0 \text{ to } x = a,$$

$$\text{and } u = a, \text{ from } x = a \text{ to } x = \tfrac{1}{2}\pi.$$

Now, since $\sin(2n+1)x = \sin(2n+1)(\pi - x)$, the values of u are the same for x as for $\pi - x$. Hence

$$u = x, \text{ from } x = 0 \text{ to } x = a,$$

$$u = a, \text{ from } x = a \text{ to } x = \pi - a,$$

$$\text{and } u = \pi - x, \text{ from } x = \pi - a \text{ to } x = \pi.$$

I have examined the other series given by Fourier, on this subject, and they seem to be all correct, with the exception of misprints and mistakes in transcription, which, unfortunately, are very numerous. The only one of these mistakes which can cause any serious embarrassment, is with regard to the series at the top of p. 245. The value of this series, between the limits $x = 0$ and $x = a$, is $\sin\frac{\pi x}{a}$, and not $\sin x$, as is there stated. The same error occurs in p. 444, where an application of this series is made to determine the value of the definite integral,

$$\int_0^\infty \frac{dq \sin qx \sin q\pi}{1 - q^2};$$

but Fourier must have been in possession of the correct value, as it is that value which he employs in this application.

P. Q. R.

VI.—ON CERTAIN INTEGRAL TRANSFORMATIONS.

By B. Bronwin.

Denby, Oct. 12th, 1840.

Sir,—I transmit you a few Integral Transformations. The integrals are reduced to elliptic, or such as are known to be elliptic functions. I am not aware that they have ever been reduced to such before, with the exception of three or four of them; and the mode of reduction is so simple as to deserve notice. Their form in general is the most useful, and the most conspicuous to serve as way-marks. Many more, however, are wanting to complete the list, so as to include every variation and combination of exponent. If that could be done they would deserve to be tabulated, with the substitutions proper to effect their reduction.

I am, Sir, your obedient servant,

B. BRONWIN.

To the Editor of the Cambridge Mathematical Journal.

In what follows, the letters E, F, denote elliptic functions, or rather functions, which by methods indicated by M. Legendre, are easily reduced to such. The letters *a*, *b*, *c*, &c. indicate the particular transformations to which they are attached.

$$\text{Let} \quad y = \frac{(1+c)\,x}{1+cx^2}, \quad k = \frac{2\sqrt{c}}{1+c} \ \ldots\ldots\ (a).$$

Then

$$\sqrt{(1-y^2)} = \frac{\sqrt{(1-x^2)}\cdot\sqrt{(1-c^2x^2)}}{1+cx^2}, \quad \sqrt{(1-k^2y^2)} = \frac{1-cx^2}{1+cx^2},$$

$$cx^2 = \frac{1-\sqrt{(1-k^2x^2)}}{1+\sqrt{(1-k^2x^2)}} = \frac{\{1-\sqrt{(1-k^2y^2)}\}^2}{k^2y^2}, \quad x = \frac{1-\sqrt{(1-k^2y^2)}}{c^{\frac{1}{2}}ky},$$

$$\sqrt{x} = \frac{\sqrt{(1+ky)}-\sqrt{(1-ky)}}{\sqrt{(2c^{\frac{1}{2}}k)}\cdot\sqrt{y}},$$

$$\frac{1}{\sqrt{x}} = \frac{\sqrt[4]{c}}{\sqrt{(2k)}}\cdot\frac{\sqrt{(1+ky)}+\sqrt{(1-ky)}}{\sqrt{y}}$$

$$dy = \frac{(1+c)\,(1-cx^2)\,dx}{(1+cx^2)^2},$$

$$\frac{dy}{\sqrt{(1-y^2)}\cdot\sqrt{(1-k^2y^2)}} = \frac{(1+c)\,dx}{\sqrt{(1-x^2)}\cdot\sqrt{(1-c^2x^2)}}.$$

From this last, by means of the equations preceding, we easily deduce the four following:

$$\frac{2\sqrt{c}\,\sqrt{(1+c)}\cdot\sqrt{x}\,dx}{\sqrt{(1-x^2)}\cdot\sqrt{(1-c^2x^2)}} = \frac{dy}{\sqrt{(1-y^2)}\,\sqrt{(y-ky^2)}} - \frac{dy}{\sqrt{(1-y^2)}\cdot\sqrt{(y+ky^2)}} \ldots\ldots(1),$$

$$\frac{2\sqrt{(1+c)}\,.\,dx}{\sqrt{x}\,\sqrt{(1-x^2)}\,.\,\sqrt{(1-c^2x^2)}} = \frac{dy}{\sqrt{(1-y^2)}.\sqrt{(y-ky^2)}} + \frac{dy}{\sqrt{(1-y^2)}.\sqrt{1+ky^2)}} \ldots\ldots(2),$$

$$\frac{2ck\sqrt{(1+c)}.x^{\frac{3}{2}}\,dx}{\sqrt{(1-x^2)}\,.\,\sqrt{(1-c^2x^2)}} = \frac{(2-ky)\,dy}{y\sqrt{(1-y^2)}.\sqrt{(y-ky^2)}} - \frac{(2+ky)\,dy}{y\sqrt{(1-y^2)}.\sqrt{(1+ky^2)}} \ldots\ldots(3),$$

$$\frac{2k\sqrt{\left(\frac{1+c}{c}\right)}\,.\,dx}{x\sqrt{x}\,\sqrt{(1-x^2)}\,.\,\sqrt{(1-c^2x^2)}} = \frac{(2-ky)\,dy}{y\sqrt{(1-y^2)}.\sqrt{(y-ky^2)}} + \frac{(2+ky)\,dy}{y\sqrt{(1-y^2)}\,\sqrt{(y+ky^2)}} \ldots\ldots(4).$$

These are all E, F. We now proceed to another transformation.

Let $y = \dfrac{\sqrt{(1-x^2)}}{\sqrt{(1-c^2x^2)}}$, or $x = \dfrac{\sqrt{(1-y^2)}}{\sqrt{(1-c^2y^2)}}$ ..... (*b*).

Then $\sqrt{(1-c^2x^2)} = \dfrac{\sqrt{(1-c^2)}}{\sqrt{(1-c^2y^2)}}$, $\sqrt{(1-x^2)} = \dfrac{y\sqrt{(1-c^2)}}{\sqrt{(1-c^2y^2)}}$,

$$\frac{x\sqrt{(1-c^2)}}{\sqrt{(1-x^2)}} = \frac{\sqrt{(1-y^2)}}{y};$$

$$dy = \frac{-(1-c^2)\,x\,dx}{(1-x^2)^{\frac{1}{2}}\,(1-c^2x^2)^{\frac{1}{2}}},$$

$$\frac{dy}{\sqrt{(1-y^2)}.\sqrt{(1-c^2y^2)}} = \frac{-dx}{\sqrt{(1-x^2)}.\sqrt{(1-c^2x^2)}}.$$

In the same manner as before we may deduce from these the following:

$$\frac{dx}{(1-x^2)^{\frac{1}{4}}\,(1-c^2x^2)^{\frac{3}{4}}} = \frac{-\sqrt{y}\,.\,dy}{\sqrt{(1-y^2)}\,.\,\sqrt{(1-c^2y^2)}} \ldots\ldots\ldots(5).$$

The last is an E, F, by (1).

$$\frac{dx}{(1-x^2)^{\frac{3}{4}}\,(1-c^2x^2)^{\frac{1}{4}}} = \frac{-dy}{\sqrt{y}\,\sqrt{(1-y^2)}\,.\,\sqrt{(1-c^2y^2)}} \ldots\ldots(6).$$

This is a E, F, by (2).

$$\frac{dx}{(1-x^2)^{\frac{1}{4}}\,(1-c^2x^2)^{\frac{1}{4}}} = \frac{-\sqrt{(1-c^2)}.\sqrt{y}\,dy}{(1-c^2y^2)\,\sqrt{(1-y^2)}} \ldots\ldots\ldots\ldots(7),$$

$$\frac{dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}} = \frac{-dy}{\sqrt{(1-c^2)}\cdot\sqrt{y}\,\sqrt{(1-y^2)}} \quad \ldots\ldots (8).$$

The two last are plainly E, F.

$$\frac{\sqrt{x}\,dx}{(1-x^2)^{\frac{1}{2}}(1-c^2x^2)^{\frac{1}{4}}} = \frac{-\sqrt[4]{(1-c^2)}\cdot dy}{(1-y^2)^{\frac{1}{4}}(1-c^2y^2)} \quad \ldots\ldots\ldots (9),$$

$$\frac{\sqrt{x}\,dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{2}}} = \frac{-\sqrt[4]{(1-c^2)}\cdot\sqrt{y}\,dy}{(1-y^2)^{\frac{1}{4}}(1-c^2y^2)} \quad \ldots\ldots\ldots\ldots (10).$$

In (9) make $(1-y^2)^{\frac{1}{4}} = v$; and in (10), $\dfrac{\sqrt{y}}{(1-y^2)^{\frac{1}{4}}} = \dfrac{1}{v}$, or $\left(\dfrac{1}{y^2}-1\right)^{\frac{1}{4}} = v$; and these are both evidently E, F.

$$\frac{\sqrt[4]{(1-c^2)}\cdot dx}{\sqrt{x}\,(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{1}{2}}} = \frac{-dy}{\sqrt{y}\,(1-y^2)^{\frac{3}{4}}} \quad \ldots\ldots\ldots (11).$$

If we make $y = \dfrac{1-v^2}{1+v^2}$, this is seen to be an E, F.

Next, let $y = \left(\dfrac{1+c}{2}\right)^{\frac{1}{2}} \dfrac{\sqrt{(1+x)}}{\sqrt{(1+cx)}}$, $\quad k = \dfrac{2\sqrt{(c)}}{1+c}$ ..... (c).

Then $\sqrt{(1-y^2)} = \left(\dfrac{1-c}{2}\right)^{\frac{1}{2}} \cdot \dfrac{\sqrt{(1-x)}}{\sqrt{(1+cx)}}$,

$$\sqrt{(1-k^2y^2)} = \left(\frac{1-c}{1+c}\right)^{\frac{1}{2}} \cdot \frac{\sqrt{(1-cx)}}{\sqrt{(1+cx)}};$$

$$\frac{dy}{\sqrt{(1-y^2)}\cdot\sqrt{(1-k^2y^2)}} = \frac{1+c}{2}\,\frac{dx}{\sqrt{(1-x^2)}\cdot\sqrt{(1-c^2x^2)}}.$$

From hence we derive, by the same process as before,

$$\frac{dx}{(1-x)^{\frac{1}{2}}(1-cx)^{\frac{1}{2}}(1+x)^{\frac{1}{4}}(1+cx)^{\frac{3}{4}}} = \left(\frac{2}{1+c}\right)^{\frac{5}{4}} \frac{\sqrt{y}\,dy}{\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}} \quad \ldots\ldots\ldots (12),$$

$$\frac{dx}{(1-x)^{\frac{1}{2}}(1-cx)^{\frac{1}{2}}(1+x)^{\frac{3}{4}}(1+cx)^{\frac{1}{4}}} = \left(\frac{2}{1+c}\right)^{\frac{3}{4}} \frac{dy}{\sqrt{y}\,\sqrt{(1-y^2)}\,\sqrt{(1-k^2y^2)}} \quad \ldots\ldots\ldots (13).$$

The two last are E, F, by (1) and (2) respectively.

$$\frac{dx}{(1-x)^{\frac{1}{4}}(1-cx)^{\frac{3}{4}}(1+x)^{\frac{1}{2}}(1+cx)^{\frac{1}{2}}} = \left(\frac{2}{1+c}\right)^{\frac{5}{4}} \cdot \frac{dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{3}{4}}} \quad \ldots\ldots\ldots (14),$$

$$\frac{dx}{(1-x)^{\frac{3}{4}}(1-cx)^{\frac{1}{4}}(1+x)^{\frac{1}{2}}(1+cx)^{\frac{1}{2}}} = \left(\frac{2}{1+c}\right)^{\frac{3}{4}} \cdot \frac{dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{4}}} \quad \ldots\ldots\ldots (15).$$

The two last are E, F, by (5) and (6).

$$\frac{dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{2}}(1+cx)} = \left(\frac{2}{1-c}\right)^{\frac{1}{4}}\left(\frac{2}{1+c}\right)^{\frac{5}{4}} \cdot \frac{\sqrt{y}\,dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots (16).$$

This is an E, F, by (10).

In all the transformations (c) we may make x negative, or

$$y = \left(\frac{1+c}{2}\right)^{\frac{1}{2}} \cdot \frac{\sqrt{(1-x)}}{\sqrt{(1-cx)}}.$$

These will give us an additional number of E, F. I shall not put them down.

Next, let

$$y = \frac{1+\sqrt{c}}{\sqrt{\{2(1+c)\}}} \cdot \frac{\sqrt{(1+x)}.\sqrt{1+cx)}}{1+c^{\frac{1}{2}}x}, \quad c = \left\{\frac{1-\sqrt[4]{(1-k^2)}}{1+\sqrt[4]{(1-k^2)}}\right\}^2 \ldots(d).$$

Then $$\sqrt{(1-y^2)} = \frac{1-\sqrt{c}}{\sqrt{\{2(1+c)\}}} \cdot \frac{\sqrt{(1-x)}\cdot\sqrt{(1-cx)}}{1+c^{\frac{1}{2}}x},$$

$$\sqrt{(1-k^2y^2)} = (1-k^2)^{\frac{1}{4}}\frac{1-c^{\frac{1}{2}}x}{1+c^{\frac{1}{2}}x}, \quad c^{\frac{1}{2}}x = \frac{\sqrt[4]{(1-k^2)}-\sqrt{(1-k^2y^2)}}{\sqrt[4]{(1-k^2)}+\sqrt{(1-k^2y^2)}};$$

$$\frac{dy}{\sqrt{(1-y^2)}\cdot\sqrt{(1-k^2y^2)}} = \frac{(1+\sqrt{c})^2}{2} \cdot \frac{dx}{\sqrt{(1-x^2)}.\sqrt{(1-c^2x^2)}}.$$

From these we derive

$$\frac{dx}{(1+x)^{\frac{1}{4}}(1-x)^{\frac{1}{2}}(1+cx)^{\frac{1}{4}}(1-cx)^{\frac{1}{2}}(1+c^{\frac{1}{2}}x)^{\frac{1}{2}}} = \frac{2\sqrt[4]{\{2(1+c)\}}}{(1+\sqrt{c})^{\frac{5}{2}}} \cdot \frac{\sqrt{y}\,dy}{\sqrt{(1-y^2)}\,\sqrt{(1-k^2y^2)}} \quad \ldots\ldots (17).$$

This is an E, F, by (1) and (2).

$$\frac{dx}{(1+x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}(1+c^{\frac{1}{2}}x)} = \frac{2\sqrt{\{2(1+c)\}}}{(1+\sqrt{c})^2\sqrt{(1-c)}} \cdot \frac{\sqrt{y}\,dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{1}{2}}} \quad \ldots\ldots (18),$$

an E, F, by (10).

By this transformation we might obtain more formulæ.

$$\text{Let} \quad y=\frac{\sqrt{(1-x^2)}\,\sqrt{(1-c^2x^2)}}{1-c^2x^2}, \quad k=\frac{2\sqrt{c}}{1+c} \;\ldots\ldots (e).$$

$$\text{Then} \quad \sqrt{(1-y^2)}=\frac{(1-c)\,x}{1-cx^2}, \quad \sqrt{(1-k^2y^2)}=\frac{1-c}{1+c}\cdot\frac{1+cx^2}{1-cx^2}.$$

$$\frac{dy}{\sqrt{(1-y^2)}\,\sqrt{(1-k^2y^2)}}=\frac{-(1+c)\,dx}{\sqrt{(1-x^2)}\,\sqrt{(1-c^2x^2)}}.$$

Hence we find

$$\frac{(1+c)\,dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}(1-cx^2)^{\frac{1}{2}}}=\frac{-\sqrt{y}\,dy}{\sqrt{(1-y^2)}\,\sqrt{(1-k^2y^2)}} \;\ldots\ldots (19),$$

$$\frac{(1+c)\,(1-cx^2)^{\frac{1}{2}}\,dx}{(1-x^2)^{\frac{3}{4}}\,(1-c^2x^2)^{\frac{3}{4}}}=\frac{-dy}{\sqrt{y}\,\sqrt{(1-y^2)}\,\sqrt{(1-k^2y^2)}} \;\ldots\ldots\ldots (20).$$

VII.—ON A THEOREM IN THE GEOMETRY OF POSITION.*

We propose to apply the following (new?) theorem to the solution of two problems in Analytical Geometry.

Let the symbols

$$|\,a\,|, \quad \begin{vmatrix} a, & \beta \\ a', & \beta' \end{vmatrix}, \quad \begin{vmatrix} a, & \beta, & \gamma \\ a', & \beta', & \gamma' \\ a'', & \beta'', & \gamma'' \end{vmatrix}, \text{ \&c.}$$

denote the quantities

$$a, \; a\beta'-a'\beta, \; a\beta'\gamma''-a\beta''\gamma'+a'\beta''\gamma-a'\beta\gamma''+a''\beta\gamma'-a''\beta'\gamma, \text{ \&c.}$$

(The law of whose formation is tolerably well known, but may be thus expressed,

$$|\,a\,|=a, \quad \begin{vmatrix} a, & \beta \\ a', & \beta' \end{vmatrix}=a\,|\,\beta'\,|-a'\,|\,\beta\,|,$$

$$\begin{vmatrix} a, & \beta, & \gamma \\ \omega', & \beta', & \gamma' \\ a'', & \beta'', & \gamma'' \end{vmatrix}=a\begin{vmatrix} \beta', & \gamma' \\ \beta'', & \gamma'' \end{vmatrix}+a'\begin{vmatrix} \beta'', & \gamma'' \\ \beta, & \gamma \end{vmatrix}+a''\begin{vmatrix} \beta, & \gamma \\ \beta', & \gamma' \end{vmatrix}, \text{ \&c.}$$

the signs + being used when the number of terms in the side of the square is odd, and + and − alternately when it is even.)

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* From a Correspondent.

Then the theorem in question is

$$\begin{vmatrix} \rho\alpha+\sigma\beta+\tau\gamma\ldots, & \rho\alpha'+\sigma\beta'+\tau\gamma'\ldots, & \rho\alpha''+\sigma\beta''+\tau\gamma''\ldots \\ \rho'\alpha+\sigma'\beta+\tau'\gamma\ldots, & \rho'\alpha'+\sigma'\beta'+\tau'\gamma'\ldots, & \rho'\alpha''+\sigma'\beta''+\tau'\gamma''\ldots \\ \rho''\alpha+\sigma''\beta+\tau''\gamma\ldots, & \rho''\alpha'+\sigma''\beta'+\tau''\gamma'\ldots, & \rho''\alpha''+\sigma''\beta''+\tau''\gamma''\ldots \\ \vdots & \vdots & \vdots \end{vmatrix}$$

$$= \begin{vmatrix} \rho, & \sigma, & \tau\ldots \\ \rho', & \sigma', & \tau'\ldots \\ \rho'', & \sigma'', & \tau''\ldots \\ \vdots & \vdots & \vdots \end{vmatrix} \cdot \begin{vmatrix} \alpha, & \beta, & \gamma\ldots \\ \alpha', & \beta', & \gamma'\ldots \\ \alpha'', & \beta'', & \gamma''\ldots \\ \vdots & \vdots & \vdots \end{vmatrix}$$

(This theorem admits of a generalisation which we shall not have occasion to make use of, and which therefore we may notice at another opportunity.)

To find the relation that exists between the distances of five points in space.

We have, in general, whatever x_1, y_1, z_1, w_1, &c., denote.

$$\begin{vmatrix} x_1^2+y_1^2+z_1^2+w_1^2, & -2x_1, & -2y_1, & -2z_1, & -2w_1, & 1. \\ x_2^2+y_2^2+z_2^2+w_2^2, & -2x_2, & -2y_2, & -2z_2, & -2w_2, & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_5^2+y_5^2+z_5^2+w_5^2, & -2x_5, & -2y_5, & -2z_5, & -2w_5, & 1 \\ 1, & 0, & 0, & 0, & 0, & 0 \end{vmatrix}$$

multiplied into

$$\begin{vmatrix} 1, & x_1, & y_1, & z_1, & w_1, & x_1^2+y_1^2+z_1^2+w_1^2 \\ 1, & x_2, & y_2, & z_2, & w_2, & x_2^2+y_2^2+z_2^2+w_2^2 \\ \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot \\ \cdot, & \cdot, & \cdot, & \cdot, & \cdot, & \cdot \\ 1, & x_5, & y_5, & z_5, & w_5, & x_5^2+y_5^2+z_5^2+w_5^2 \\ 0, & 0, & 0, & 0, & 0, & 1 \end{vmatrix} =$$

$$\begin{vmatrix} \overline{\quad}^2 & \overline{\quad}^2 & \overline{\quad}^2 & \overline{\quad}^2 & & \\ x_1-x_1+y_1-y_1+z_1-z_1+w_1-w_1, & & & & & \\ & \overline{\quad}^2 & \overline{\quad}^2 & \overline{\quad}^2 & \overline{\quad}^2 & \\ & x_1-x_2+\ldots, & x_1-x_3+\ldots, & x_1-x_4+\ldots, & x_1-x_5+\ldots, & 1 \\ \overline{\quad}^2 & & & & & \\ x_2-x_1+\ldots & , & & & & \\ & \overline{\quad}^2 & \overline{\quad}^2 & \overline{\quad}^2 & \overline{\quad}^2 & \\ & x_2-x_2+\ldots, & x_2-x_3+\ldots, & x_2-x_4+\ldots, & x_2-x_5+\ldots, & 1 \\ \cdot & \cdot & , \cdot & , \cdot & , \cdot & , \cdot \\ \cdot & \cdot & , \cdot & , \cdot & , \cdot & , \cdot \\ \overline{\quad}^2 & & & & & \\ x_5-x_1\ldots & , & & & & \\ & \overline{\quad}^2 & \overline{\quad}^2 & \overline{\quad}^2 & \overline{\quad}^2 & \\ & x_5-x_2+\ldots, & x_5-x_3+\ldots, & x_5-x_4+\ldots, & x_5-x_5+\ldots, & 1 \\ & 1 & , 1 & , 1 & , 1 & , 0 \end{vmatrix}$$

Putting the w's equal to 0, each factor of the first side of the equation vanishes, and therefore in this case the second side of the equation becomes equal to zero. Hence $x_1, y_1, z_1, x_2, y_2, z_2$, &c. being the co-ordinates of the points 1, 2, &c. situated arbitrarily in space, and $\overline{12}^2, \overline{13}^2$, &c. denoting the distances between these points, we have immediately the required relation

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2, & 1 \\ \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

which is easily expanded, though from the mere number of terms the process is somewhat long.

Precisely the same investigation is applicable to the case of four points in a plane, or three points in a straight line. Thus the former gives

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & 1 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

The latter gives

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & 1 \\ \overline{21}^2, & 0, & \overline{23}^2, & 1 \\ \overline{31}^2, & \overline{32}^2, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0.$$

Or expanding,

$$\overline{12}^4 + \overline{13}^4 + \overline{23}^4 - 2\,\overline{12}^2\overline{13}^2 - 2\,\overline{13}^2\overline{23}^2 - 2\,\overline{12}^2\overline{23}^2 = 0;$$

which may be derived immediately, by the equation

$$\pm\sqrt{\overline{12}} \pm \sqrt{\overline{13}} = \pm\sqrt{\overline{23}},$$

and is the simplest form under which this equation, cleared of the ambiguous sign, can be put.

(The above result may be deduced so elegantly from the general theory of elimination, that notwithstanding its simplicity it is perhaps worth mentioning.)

Let $x_{//} - x_{///} = a, \quad x_{///} - x_{/} = \beta, \quad x_{/} - x_{//} = \gamma;$

then $\overline{12}^2 = \gamma^2, \quad \overline{23}^2 = a^2, \quad \overline{31}^2 = \beta^2,$ and $a + \beta + \gamma = 0;$

from which $a, \beta, \gamma,$ are to be eliminated. Multiplying the last equation by $\beta\gamma, \gamma a, a\beta,$ and reducing by the three first,

$$\begin{aligned}
0 \,.\, a + \overline{12}^2 \,.\, \beta + \overline{31}^2 \,.\, \gamma + a\beta\gamma &= 0, \\
\overline{12}^2 \,.\, a + 0 \,.\, \beta + \overline{23}^2 \,.\, \gamma + a\beta\gamma &= 0, \\
\overline{31}^2 \,.\, a + \overline{23}^2 \,.\, \beta + 0 \,.\, \gamma + a\beta\gamma &= 0, \\
a + \beta + \gamma + 0 \,.\, a\beta\gamma &= 0;
\end{aligned}$$

from which, eliminating $a, \beta, \gamma, a\beta\gamma$ by the general theory of simple equations,

$$\begin{vmatrix}
0, & \overline{12}^2, & \overline{13}^2, & 1 \\
\overline{21}^2, & 0, & \overline{23}^2, & 1 \\
\overline{31}^2, & \overline{32}^2, & 0, & 1 \\
1, & 1, & 1, & 0
\end{vmatrix} = 0.$$

The (additional) equation that exists between the distances of five points on a sphere or four points in a circle, has such a remarkable analogy with the preceding, that they almost require to be noticed at the same time.

If a, β, γ, r be the co-ordinates of the centre, and the radius of the sphere, and

$$\delta = a^2 + \beta^2 + \gamma^2 - r^2,$$

we have immediately

$$\begin{aligned}
x_1^2 + y_1^2 + z_1^2 - 2ax_1 - 2\beta y_1 - 2\gamma z_1 + \delta &= 0, \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
x_5^2 + y_5^2 + x_5^2 - 2ax_5 - 2\beta y_5 - 2\gamma z_5 + \delta &= 0;
\end{aligned}$$

whence eliminating $a, \beta, \gamma, \delta,$

$$\begin{vmatrix}
x_1^2 + y_1^2 + z_1^2, & -2x_1, & -2y_1, & -2z_1, & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_5^2 + y_5^2 + z_5^2, & -2x_5, & -2y_5, & -2z_5, & 1
\end{vmatrix} = 0.$$

Multiplying by

$$\begin{vmatrix}
1, & x_1, & y_1, & z_1, & x_1^2 + y_1^2 + z_1^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1, & x_5, & y_5, & z_5, & x_5^2 + y_5^2 + z_5^2
\end{vmatrix}$$

we have immediately

$$\begin{vmatrix} 0, & \overline{12}^2, & \overline{13}^2, & \overline{14}^2, & \overline{15}^2 \\ \overline{21}^2, & 0, & \overline{23}^2, & \overline{24}^2, & \overline{25}^2 \\ \overline{31}^2, & \overline{32}^2, & 0, & \overline{34}^2, & \overline{35}^2 \\ \overline{41}^2, & \overline{42}^2, & \overline{43}^2, & 0, & \overline{45}^2 \\ \overline{51}^2, & \overline{52}^2, & \overline{53}^2, & \overline{54}^2, & 0 \end{vmatrix} = 0;$$

forming the analogous equation for four points in a circle, and expanding, we readily deduce

$$\overline{14}^4 . \overline{23}^4 + \overline{12}^4 \overline{34}^4 + \overline{13}^4 . \overline{24}^4 - 2 . \overline{12}^2 \overline{34}^2 \overline{13}^2 \overline{24}^2 - 2 . \overline{14}^2 \overline{23}^2 \overline{13}^2,$$

$$\overline{24}^2 - 2 . \overline{14}^2 \overline{23}^2 \overline{12}^2 \overline{34}^2 = 0,$$

which is the rational, and therefore analytically the most simple form of

$$\overline{12}\ \overline{34} + \overline{14}\ \overline{23} = \overline{13}\ \overline{24}.$$

Euclid, B. vi., last proposition.

(It may be remarked that the two factors we have employed in the preceding eliminations, only differ by a numerical factor.)

A. Cayley

VIII.—ANALYTICAL DEMONSTRATIONS OF DR. MATTHEW STEWART'S THEOREMS.

By R. Leslie Ellis, B.A., Fellow of Trinity College.

In 1746, Dr. Matthew Stewart, the father of Dugald Stewart, published his "General Theorems." He was at that time a candidate for the chair of mathematics at Edinburgh, then vacant by the death of Maclaurin; and his success is attributed to the celebrity which these remarkable propositions immediately acquired. They were enunciated by Dr. Stewart without demonstrations, and remained undemonstrated till 1805. Mr. Glenie, in the *Edinburgh Transactions* for that year, has given a geometrical method by which the General Theorems and other similar results may be established.

But as yet they have not, I believe, been proved, except by Geometry; and in an article in the 17th volume of the *Edinburgh Review*, ascribed to Playfair, they are strongly recommended to

the attention of analysts. It is hoped, therefore, that the following attempt will have some degree of interest.

We shall begin by establishing a general proposition, from which all the theorems in question, and many others, may be deduced.

LEMMA. If $f\phi$ is a rational and integral function of sin ϕ and cos ϕ, then a value may always be assigned to n, such that

$$f(\phi) + f\left(\phi + \frac{2\pi}{n}\right) + \ldots f\left(\phi + \frac{n-1}{n}\,2\pi\right)$$

shall be independent of ϕ.

The preceding expression is equivalent to

$$(1 + \mathrm{D} + \ldots \mathrm{D}^{n-1}) f\phi, \quad \left(\text{where } \mathrm{D}\phi = \phi + \frac{2\pi}{n}\right),$$

and therefore to

$$\frac{\mathrm{D}^n f\phi - f\phi}{\mathrm{D} - 1} = \Delta^{-1}\{f(\phi + 2\pi) - f\phi\} = \Delta^{-1} 0.$$

Now $\Delta^{-1} 0 = \Sigma a_m \sin mn\phi + \Sigma b_m \cos mn\phi$, ($m$ integral).

Hence $f(\phi) + \ldots + f\left(\phi + \frac{n-1}{n}\,2\pi\right) = \Sigma a_m \sin mn\phi +$ &c.

Let the index of the highest power of sin ϕ or cos ϕ in $f\phi$ be p; then it is easily seen that when $f\phi$ is developed, as it may always be, in a series of sines and cosines of the multiple arcs, $p\phi$ will be the largest arc that can enter into the development. But if n is greater than p, $mn\phi$ will be greater than $p\phi$, except when m is zero. Hence the development

$$\Sigma a_m \sin mn\phi + \text{&c.}$$

cannot coincide with that obtained by summing the separate developments of $f\phi$, $f\left(\phi + \frac{2\pi}{n}\right)$, &c., unless a_m and b_m are $= 0$ in every case, except when $m = 0$. Hence as sin $mn\phi = 0$ when $m = 0$, the expression will be reduced to b_0, and we shall have, when $n > p$,

$$f(\phi) + \ldots f\left(\phi + \frac{n-1}{n}\,2\pi\right) = b_0 \ldots \text{a constant.}$$

Q. E. D.

The constant b_0, will of course be the sum of the constant parts of the developments of $f\phi$, &c.; and as these are all equal and are n in number, it will be n times the constant term in $f\phi$. Now by Fourier's theorem this is equal to $\frac{1}{2\pi}\int_0^{2\pi} f\phi \,.\, d\phi$, as indeed is obvious. Hence

$$f\phi + \ldots f\left(\phi + \frac{n-1}{n}\,2\pi\right) = \frac{n}{2\pi}\int_0^{2\pi} f\phi \, d\phi, \text{ when } n > p,$$

which is our fundamental formula.

The first of Stewart's propositions is the following:

From any point in the circumference of a circle draw perpendiculars p, p_1, &c. to the sides of a regular n-sided polygon circumscribed about it; then, if r is the radius,

$$2\Sigma p^3 = 5nr^3 \ldots\ldots \ldots\ldots (1) \ldots\ldots\ldots n > 3.$$

Dem. Let the assumed point subtend at the centre an angle ϕ from the adjacent point of contact. Then

$$p = r(1-\cos\phi), \quad p_1 = r\left\{1 - \cos\left(\phi + \frac{2\pi}{n}\right)\right\}, \text{ &c. &c.}$$

$$\therefore\ \Sigma p^3 = r^3 \Sigma (1-\cos\phi)^3;$$

and by the general formula,

$$\Sigma(1-\cos\phi)^3 = \frac{n}{2\pi}\int_0^{2\pi} (1-\cos\phi)^3\, d\phi \ldots\ldots n \text{ being } > 3.$$

Now $$\int_0^{2\pi} (1-\cos\phi)^3\, d\phi = 2^4 \int_0^{\pi} \sin^6\theta\, d\theta \,.. (\theta = \tfrac{1}{2}\phi)$$

and $$\int_0^{\pi} \sin^6\theta\, d\theta = \frac{5.3.1}{6.4.2}\cdot\pi = \frac{5}{2^5}\,2\pi;$$

$$\therefore\ \Sigma(1-\cos\phi)^3 = \frac{5n}{2},$$

and therefore $2\Sigma p^3 = 5nr^3$. Q. E. D.

This is a particular case of the second proposition in which the assumed point is not confined to the circumference of the circle, but may have any position whatever. Let l be its distance from the centre; then

$$2\Sigma p^3 = 2nr^3 + 3nl^2 r \ldots n > 3 \ldots\ldots\ldots (2).$$

Dem. In this case $p = r - l\cos\phi$, &c. = &c.

$$\therefore\ \Sigma p^3 = nr^3 - 3r^2 l \Sigma\cos\phi + 3rl^2 \Sigma\cos^2\phi - l^3 \Sigma\cos^3\phi.$$

But $$\int_0^{2\pi} \cos\phi\, d\phi = 0, \quad \int_0^{2\pi} \cos^2\phi\, d\phi = \pi, \quad \int_0^{2\pi} \cos^3\phi\, d\phi = 0;$$

$$\therefore\ \Sigma\cos\phi = 0, \quad \Sigma\cos^2\phi = \frac{n}{2}, \quad \Sigma\cos^3\phi = 0,$$

and $2\Sigma p^3 = 2nr^3 + 3nl^2 r$. Q. E. D.

In the third proposition, a regular n-sided polygon is *inscribed* in the circle, and lines c, c_1, &c. are drawn from its corners to a point assumed in the circumference; then

$$\Sigma c^4 = 6nr^4 \ldots\ldots\ldots (3).$$

Dem. The assumed point and adjacent corner subtending an angle ϕ at the centre, we have

$$c^2 = 2r^2(1-\cos\phi);$$

$$\therefore\ \Sigma c^4 = 4r^4\,\Sigma(1-\cos\phi)^2,$$

$$\int_0^{2\pi}(1-\cos\phi)^2\,d\phi = 2^3\cdot\frac{3\,.\,1}{4\,.\,2}\,\pi = 2\pi\,\frac{3}{2};$$

$$\therefore\ \Sigma(1-\cos\phi)^2 = \frac{3n}{2};$$

$$\therefore\ \Sigma c^4 = 6nr^4. \qquad \text{Q. E. D.}$$

The fourth proposition includes the third. The assumed point may now have any position we please. Let l be its distance from the centre. Here we have

$$c^2 = r^2 + l^2 - 2rl\cos\phi,$$

and $\Sigma c^4 = n(r^4 + l^4 + 2r^2l^2) - 4rl(r^2 + l^2)\,\Sigma\cos\phi + 4r^2l^2\,\Sigma\cos^2\phi.$

By the values above given for $\Sigma\cos\phi$ and $\Sigma\cos^2\phi$, this becomes

$$\Sigma c^4 = nr^4 + 4nr^2l^2 + nl^4 \ldots\ldots (4),$$

which is the proposition in question.

In the fifth proposition we return to the circumscribed polygon, and our object is to determine the sum of the fourth powers of the perpendiculars. As before,

$$\Sigma p^4 = r^4\,\Sigma(1-\cos\phi)^4,$$

and $$\int_0^{2\pi}(1-\cos\phi)^4\,d\phi = 2^5\,\frac{7\,.\,5\,.\,3\,.\,1}{8\,.\,6\,.\,4\,.\,2}\,\pi = 2\pi\,.\,\frac{35}{8}\,.$$

therefore $8\,\Sigma p^4 = 35n\,.\,r^4$ ..... (5). Q. E. D.

In the general case, when l is the distance of the assumed point from the centre,

$$\Sigma p^4 = nr^4 - 4r^3l\,\Sigma\cos\phi + 6r^2l^2\,\Sigma\cos^2\phi - 4rl^3\,\Sigma\cos^3\phi + l^4\,\Sigma\cos^4\phi,$$

$$\text{and}\quad \Sigma\cos^4\phi = \frac{n}{2}\cdot\frac{3\,.\,1}{4\,.\,2} = \frac{3n}{8};$$

$$\therefore\ \Sigma p^4 = nr^4 + 6r^2l^2\,.\,\frac{n}{2} + \frac{3}{8}\,nl^4,$$

$$\text{or}\quad 8\,\Sigma p^4 = 8nr^4 + 24nr^2l^2 + 3nl^4 \ldots\ldots (6).$$

This is the sixth proposition.

The seventh is, for the m^{th} power of the perpendiculars, what the first and fifth are for the 3^{d} and 4^{th} powers respectively. It is this:

$$\Sigma p^m = n\,\frac{2m-1\,.\,2m-3\ldots 1}{m\,.\,m-1\,\ldots\,1}\,r^m \ldots\ldots (7) \ldots\ldots (n>m).$$

Dem. $$\Sigma p^m = r^m\,\frac{n}{2\pi}\int_0^{2\pi}(1-\cos\phi)^m\,d\phi.$$

Let $\theta = \frac{1}{2}\phi$;

$$\therefore \int_0^{2\pi} (1-\cos\phi)^m \, d\phi = 2^{m+1} \int_0^{\pi} \sin^{2m}\theta \, d\theta$$

$$= 2^{m+1} \frac{2m-1 \,.\, 2m-3 \ldots 1}{2m \,.\, 2m-2 \ldots 2} \pi = 2\pi \,.\, \frac{2m-1 \ldots 1}{m \,.\, m-1 \ldots 1}$$

$$\therefore \Sigma p^m = n \frac{2m-1 \,.\, 2m-3 \ldots 1}{m \,.\, m-1 \ldots 1} r^m.$$

Q. E. D.

If the assumed point is at l distance from the centre,

$$\Sigma p^m = nr^m - \frac{m}{1} \,.\, r^{m-1} l \, \Sigma \cos\phi + \frac{m \,.\, m-1}{1 \,.\, 2} r^{m-2} l^2 \, \Sigma \cos^2\phi - \&c.$$

This, it is easily seen, will reduce into the following form

$$\Sigma p^m = nr^m + n \frac{m \,.\, m-1}{1.2} \frac{1}{2} r^{m-2} l^2 + n \frac{m \ldots m-3}{1.2.3.4} \frac{3.1}{4.2} r^{m-4} l^4 + \&c.$$

$$\ldots\ldots(8),$$

which is the eighth proposition.... $(n > m)$.

Lastly, let us consider the $2m^{\text{th}}$ powers of the chords in the case of the inscribed polygon: we have already in the third proposition found the value of their sum when $m = 2$.

As before, $c^2 = 2r^2 (1 - \cos\phi)$;

$$\therefore \Sigma c^{2m} = 2^m r^{2m} \Sigma (1 \quad \cos\phi)^m.$$

That is, as we have already seen,

$$\Sigma c^{2m} = n \frac{2m-1 \,.\, 2m-3 \ldots 1}{m \,.\, m-1 \ldots 1} r^{2m} \ldots \ldots (9).$$

We have thus gone through Dr. Stewart's properties of the circle, and have arrived at his results by a simple and uniform method.

It is evident that there is no limit to the number of geometrical theorems which may be deduced from the general formula: almost every curve will afford *interpretations*, if the word may be so used, of our analytical conclusions.

Thus in the ellipse: If any n radii vectores be drawn from the centre at equal angles to one another, the sum of the squares of their reciprocals is equal to n times the square of the reciprocal of that radius vector which is equally inclined to the major and minor axes. For we have

$$\frac{1}{r^2} = \frac{1}{b^2} (1 - e^2 \cos^2\phi);$$

$$\therefore \Sigma \frac{1}{r^2} = \frac{n}{b^2} - \frac{e^2}{b^2} \Sigma \cos^2\phi = n \frac{1}{b^2} (1 - \tfrac{1}{2} e^2),$$

and $\frac{1}{2} = \cos^2 \frac{\pi}{4}$;

$$\therefore\ \Sigma\frac{1}{r^2} = \frac{n}{b^2}\left(1 - e^2\cos^2\frac{\pi}{4}\right),$$

therefore &c. Q. E. D.

It is to be regretted that we have hardly any idea by what considerations Dr. Stewart was led to the curious theorems which bear his name. It is said, indeed, that he was engaged on geometrical porisms when he discovered them, and we are told that he would have published them under the title of porisms, but for his unwillingness to interfere with a subject which the researches of his friend, Dr. Simson, seemed to have appropriated. Whether they are in reality porismatic, is a question on which it would not be worth while to enter.

The fundamental formula of our analysis is perhaps not new; the geometrical applications which we have made of it appear to be original.

IX.—ON A METHOD OF ALGEBRAIC ELIMINATION.*

It is the object of this paper to explain a method of elimination common to algebraic equations, and also to differential equations of all orders and degrees. When applied to the former class, it will always, I believe, lead to calculations which do not differ much from those required by the method in common use; but in principle it appears to me much more simple and satisfactory than that method.

Let it be required to eliminate x from the equations

$$x^2 + px + q = 0,$$
$$x^2 + p'x + q' = 0.$$

Multiply each of the proposed equations by x, and you obtain

$$x^3 + px^2 + qx = 0,$$
$$x^3 + p'x^2 + q'x = 0.$$

These two combined with the two given equations make a system of four equations containing three quantities to be eliminated, viz. x, x^2, x^3; and they are of the first degree with respect to each of these quantities. We may therefore eliminate x, x^2, and x^3, by the rules for equations of the first degree. The result is

$$pq' - p'q + \frac{(q-q')^2}{p-p'} = 0.$$

The same result may be obtained rather more simply thus.

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* From a Correspondent.

Subtract the second equation from the first, and you obtain

$$(p-p')\,x + q - q' = 0.$$

Multiply this equation by x, and the first by $p-p'$, then subtract, and the resulting equation is

$$\{p\,(p-p') - (q-q')\}\,x + q\,(p-p') = 0.$$

Eliminate x between this and the other equation of the first degree; and you obtain

$$(q-q')\,\{p\,(p-p') - (q-q')\} - q\,(p-p')^2 = 0,$$

or

$$(p-p')\,\{p\,(q-q') - q\,(p-p')\} - (q-q')^2 = 0.$$

or, as before,

$$p'q - pq' - \frac{(q-q')^2}{p-p'} = 0.$$

It is evident that the methods applied to this example may be employed in treating equations of any degree, and the following rules express both the principle and the method of eliminating one unknown quantity between two algebraic equations which exceed the first degree.

First, suppose the two equations to be of the same degree; and let x denote the quantity to be eliminated, and n the degree of the equations.

Rule. "Multiply each equation by x, n times successively. You thus obtain $2n$ new equations Combine these with whichever you please of the proposed equations, and you have $2n+1$ equations, containing $2n$ quantities to be eliminated, viz. x, x^2, $x^3 \ldots x^{2n}$, and being of the first degree with regard to these quantities. From this system of equations eliminate all the $2n$ quantities by the rules for equations of the first degree."

In practice, the following rule, which is founded on the same principle as the preceding, may frequently be found more convenient. "First eliminate x^n between the two proposed equations. Then multiply the equation thus obtained by x, and eliminate x^n between the resulting equation and either (say the first) of the proposed. You will now have two equations of the $(n-1)^{\text{th}}$ degree. Treat these as you did the two proposed equations, and you will obtain two of the $(n-2)^{\text{th}}$ degree. Continue this process till you obtain an equation not at all affected with x."

If the two proposed equations are not of the same degree, let m denote the degree of the first and n that of the second, n being $< m$.

Rule. "Multiply the second equation by x, $m-n$ times successively; and eliminate x^m, x^{m-1}, $x^{m-2} \ldots x^{n+1}$ between the equations thus formed and the first. You will then have two equations, both of the n^{th} degree, which treat by either of the former rules."

It seems unnecessary to give additional examples in illustration of these rules, as every reader can easily supply such as are likely to be most satisfactory to himself. I will therefore proceed at once to shew, that the same principle by which algebraic equations of

the higher degrees are reduced to equations of the first degree, enables us to reduce differential equations of all orders and degrees, and any number of independent variables, to algebraic equations.

Ex. 1. Let it be required to eliminate y from the equations

$$\frac{dx}{dt} + 4x + 3y = t,$$

$$2x + \frac{dy}{dt} + 5y = e^t.$$

(See Vol. I. p. 175, of this Journal.)

Differentiate the first equation with regard to t, and you obtain the following:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3\frac{dy}{dt} = 1.$$

Eliminate $\frac{dy}{dt}$ between this and the second, and the result is

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} - 6x - 15y = 1 - 3e^t.$$

Eliminate y between this last result and the first equation, and you obtain a final equation altogether free from y, viz.

$$\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 14x = 1 - 3e^t + 5t.$$

Ex. 2 Let it be required to eliminate x from the equations

$$\frac{dx}{dt} + px + q = 0,$$

$$\frac{dx}{dt} + p'x + q' = 0,$$

(p, q, p', and q' being functions of t, but not containing x).

By subtraction,

$$(p - p')\,x + (q - q') = 0.$$

Differentiate this equation with regard to t, and you obtain

$$(p - p')\frac{dx}{dt} + x\frac{d}{dt}(p - p') + \frac{d}{dt}(q - q') = 0.$$

Eliminate $\frac{dx}{dt}$ between this and the first equation, and the result is

$$\left\{p(p-p') - \frac{d}{dt}(p-p')\right\}x + q(p-p') - \frac{d}{dt}(q-q') = 0.$$

Eliminate x between the two equations which are free from $\frac{dx}{dt}$, and you obtain a final equation altogether free from x, viz.

$$\left\{p(p-p') - \frac{d}{dt}(p-p')\right\}(q-q') - (p-p')\left\{q(p-p') - \frac{d}{dt}(q \quad q')\right\} = 0,$$

or

$$(p-p')(p'q-pq')-(q-q')\frac{d}{dt}(p-p')+(p-p')\frac{d}{dt}(q-q')=0.$$

The following rules express the general method of eliminating a function from two equations which contain itself and its differential coefficients with regard to one independent variable.

First, suppose the two equations to be of the same order, and let m denote their order, y the function to be eliminated, t the independent variable.

RULE. "Differentiate both equations m times with respect to t. You thus obtain $2m$ new equations; between which and either (say the first) of the proposed you may eliminate the quantities

$$y,\ \frac{dy}{dt},\ \frac{d^2y}{dt^2},\ \ldots\ldots\ \frac{d^{2m}y}{dt^{2m}},$$

by the rules for algebraic equations."

(This rule is not new, see Lacroix, *Traité Elem. de Calc. Dif.*, Art. 134, 3rd edit.)

But it may often be more convenient to proceed by successive eliminations in the following manner:

"First, eliminate $\frac{d^my}{dt}$ between the two proposed equations; then differentiate the equation thus obtained, and between the equation which results from this differentiation and either (say the first) of the proposed, again eliminate $\frac{d^my}{dt^m}$. You will now have two equations of the $(m-1)^{\text{th}}$ order. Treat these as you did the two proposed, and you obtain two of the $(m-2)^{\text{th}}$ order; and by repeating this process as often as necessary, you at last obtain an equation altogether free from y.

If the two proposed equations are not of the same order, let m denote the order of the first, and n that of the second, n being $< m$.

RULE. "Differentiate the second equation $(m-n)$ times; and between the equations thus formed and the first of the proposed, eliminate

$$\frac{d^my}{dt^m},\ \frac{d^{m-1}y}{dt^{m-1}},\ \frac{d^{m-2}y}{dt^{m-2}},\ \ldots\ldots\ \frac{d^{n+1}y}{dt^{n+1}};$$

you will then have two equations of the n^{th} order, which may be treated by the former rule."

When the proposed equations contain, besides the function to be eliminated, its differential coefficients with regard to more than one independent variable, the number of auxiliary equations to be formed is much more numerous: but by repeating the process of differentiation sufficiently often, we always obtain at last more

equations than quantities to be eliminated. If the two proposed equations are of the first order, and there are two independent variables, there will be in the proposed equations three quantities to be eliminated, viz. u, $\frac{du}{dx}$, and $\frac{du}{dy}$, (the function being denoted by u, and the independent variables by x and y). If each of the equations be differentiated with respect to x and with respect to y, there will be in all six equations, containing six quantities, to be eliminated, viz. u, $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dx\,dy}$, $\frac{d^2u}{dy^2}$. But if we proceed to the third order, by differentiating each of the new equations with regard both to x and y, we shall have in all fourteen equations, containing only ten quantities to be eliminated, viz.

$$u,\ \frac{du}{dx},\ \frac{du}{dy},\ \frac{d^2u}{dx^2},\ \frac{d^2u}{dx\,dy},\ \frac{d^2u}{dy^2},\ \frac{d^3u}{dx^3},\ \frac{d^3u}{dx^2\,dy},\ \frac{d^3u}{dx\,dy^2},\ \frac{d^3u}{dy^3}.$$

The particular form of the proposed will, however, frequently render a much smaller number of auxiliary equations sufficient, as in the following example.

Let it be required to eliminate u from the equations

$$c\frac{du}{dx} + \frac{dz}{dx} + a\frac{dz}{dy} + bz = 0,$$

$$\frac{du}{dx} + a\frac{du}{dy} + bu + c'\frac{dz}{dx} = 0.$$

(See Vol. I., p. 180, of this Journal.)

Differentiate the first of these with respect to x and y, and the second with respect to x only, and you obtain the following equations:

$$c\frac{d^2u}{dx^2} + \frac{d^2z}{dx^2} + a\frac{d^2z}{dx\,dy} + b\frac{dz}{dx} = 0,$$

$$c\frac{d^2u}{dx\,dy} + \frac{d^2z}{dx\,dy} + \frac{d^2z}{dy^2} + b\frac{dz}{dy} = 0,$$

$$\frac{d^2u}{dx^2} + a\frac{d^2u}{dx\,dy} + b\frac{du}{dx} + c'\frac{d^2z}{dx^2} = 0.$$

The three new equations and the first of the proposed contain only three quantities to be eliminated, viz. $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dx\,dy}$; these may therefore be eliminated by the methods employed for algebraic equations. As the equations are of the first degree, the process is very simple; the result is

$$(1-cc')\frac{d^2z}{dx^2} + 2a\frac{d^2z}{dx\,dy} + a^2\frac{d^2z}{dy^2} + 2b\frac{dz}{dx} + 2ab\frac{dz}{dy} + b^2z = 0.$$

In the preceding propositions the problem of elimination seems to me to be placed in a simple and luminous point of view. The different cases treated are referred to one common principle, which seems obvious as soon as it is enunciated, and which will probably be found applicable to other classes of equations.

I shall be very glad if this contribution is seen in as favourable a light by the readers of this Journal, and is found practically useful.

A. Q. G. C.

Feb. 8th, 1841.

April 2nd, 1841.

P.S.—The preceding system of elimination was suggested to me by reading an article on Simultaneous Differential Equations, in the fourth number of this Journal. I observed that the method of separating the symbols of operation from those of quantity, employed in that article, is, so far as elimination is concerned, the same in fact as the method given by La Croix. (The method of the separation of the symbols being however applicable to none but equations of the first degree; but having the advantage, where it is applicable, of indicating at once what differentiations are necessary.) This reflection led me to observe the principle on which elimination between two differential equations depends, viz. that whereas the proposed equations contain several functions of the quantity to be eliminated, this difficulty is evaded by *forming new equations.*

The process by which these equations are formed introduces new functions; nevertheless it answers the purpose for which it is employed, because it increases the number of equations still more than that of functions.

This method seemed natural, and properly applicable to the problem; whereas the method employed in treating algebraic equations of the higher degrees had always appeared to me very unsatisfactory, and obscure in principle. The question was thus raised, whether a method similar to that used in treating differential equations might not be discovered for algebraic. This question being once asked, the answer to it was soon found; especially as the paper which had suggested these reflections pointed out an analogy between the processes of Differentiation and Multiplication.

At the time when this occurred to me, and when I sent my article to you, I was entirely ignorant that any other mathematician had been occupied with the subject, and was not aware that there was any known method of elimination between algebraic equations, except that which makes the problem depend on the method of finding the greatest common measure. From Professor Sylvester's interesting paper in your last number, and from his paper in the *Philosophical Magazine* to which you referred me, I find that that gentleman has not only anticipated me in the fundamental idea, but has likewise devised some very ingenious rules for the more expeditious, and even merely mechanical, application of it. Should you, however, think my article worth inserting,

I shall feel obliged by your allowing this postscript to accompany it, which I will conclude with a remark suggested by Mr. Sylvester's last paper.

If we have to eliminate x between the two equations

$$x^n + px^{n-1} + qx^{n-2} \ldots\ldots + u = 0,$$
$$x^n + p'x^{n-1} + q'x^{n-2} \ldots\ldots + u' = 0,$$

we may first eliminate x^n by subtraction, and then write the equations in the form

$$x(x^{n-1} + px^{n-2} + qx^{n-3} + \&c.) + u = 0,$$
$$x(x^{n-1} + p'x^{n-2} + q'x^{n-3} + \&c.) + u' = 0,$$

and eliminate the unbracketed x between these. We thus obtain two equations of the $(n-1)^{\text{th}}$ degree; and in obtaining them we treat the two proposed equations *in the same manner*, which seems preferable to the other methods.

X.—NOTE ON THE DEFINITE INTEGRAL

$$\int_0^{\frac{\pi}{2}} \log(\sin\theta)\, d\theta.$$

The value of $\int_0^{\frac{\pi}{2}} \log\sin\theta\, d\theta$, obviously the same as that of $\int_0^{\frac{\pi}{2}} \log\cos\theta\, d\theta$, was first assigned by Euler, and may be obtained in the following manner.

By Cotes's theorem,

$$z^{2m}-1=(z^2-1)\left(z^2-2z\cos\frac{1}{m}\pi+1\right)\ldots\left(z^2-2z\cos\frac{m-1}{m}\pi+1\right) \quad \ldots\ldots (1).$$

Let $z=1$, then

$$m = 2^{2(m-1)}\sin^2\frac{1}{m}\frac{\pi}{2}\ldots\ldots\sin^2\frac{m-1}{m}\frac{\pi}{2}.$$

Take the logarithms of both sides, and divide by m, then

$$\frac{\log m + 2(m-1)\log\frac{1}{2}}{2m} = \left(\log\sin\frac{1}{m}\frac{\pi}{2} + \ldots + \log\sin\frac{m-1}{m}\frac{\pi}{2}\right)\frac{1}{m}.$$

Let m become infinite, and $=\frac{1}{dx}$: the first side becomes equal

to $\log \frac{1}{2}$, for $\left(\frac{\log m}{m}\right) = 0$ when $m = \frac{1}{0}$; and the second is transformed into the definite integral $\int_0^1 \log \sin x \frac{\pi}{2} \cdot dx$; therefore $\int_0^1 \log \sin x \frac{\pi}{2}\, dx = \log \frac{1}{2}$.

Let $\theta = x \frac{\pi}{2}$;

$$\therefore \int_0^{\frac{\pi}{2}} \log \sin \theta \, d\theta = \frac{\pi}{2} \log \tfrac{1}{2} \ldots\ldots\ldots (1).$$

Cor. 1. Integrating by parts, we get

$$\int \log \sin \theta \, d\theta = \theta \log \sin \theta - \int \frac{\theta \, d\theta}{\tan \theta}.$$

The integrated part vanishes at both limits,

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\theta \, d\theta}{\tan \theta} = \frac{\pi}{2} \log 2 \ldots\ldots\ldots (2).$$

Cor. 2. Let $\sin \theta = e^{-\frac{1}{2}x}$; therefore the limits of x are 0 and ∞,

$$\text{and } d\theta = -\tfrac{1}{2} \frac{e^{-\frac{1}{2}x}}{\sqrt{(1-e^{-x})}}\, dx = -\tfrac{1}{2} \frac{dx}{\sqrt{(e^x-1)}};$$

$$\therefore \int_0^{\infty} \frac{x \, dx}{\sqrt{(e^x-1)}} = 2\pi \log 2 \ldots\ldots (3).$$

Cor. 3. In this last integral, if we expand the denominator

$$(\epsilon^x-1)^{-\frac{1}{2}} = \epsilon^{-\frac{x}{2}} + \frac{1}{2} \epsilon^{-\frac{3x}{2}} + \frac{1.3}{1.2} \frac{1}{2^2} \epsilon^{-\frac{5x}{2}} + \&c.$$

$$\text{and as } \int_0^{\infty} \epsilon^{-mx} x \, dx = \frac{1}{m^2}, \text{ we find}$$

$$\frac{\pi}{2} \log 2 = 1 + \frac{1}{1} \cdot \frac{1}{2} \frac{1}{3^2} + \frac{1.3}{1.2} \frac{1}{2^2} \frac{1}{5^2} + \frac{1.3.5}{1.2.3} \frac{1}{2^3} \frac{1}{7^2} + \&c.$$

R. L. E.

XI.—MATHEMATICAL NOTES.

The properties of Laplace's functions, of which demonstrations are given at p. 192 of this volume, may be extended to all symmetrical homogeneous functions of two variables.

Let u be a homogeneous function of n dimensions in x and y Then, if it be symmetrical, we have

$$u = x^n f\left(\frac{y}{x}\right) = y^n f\left(\frac{x}{y}\right).$$

But by a known theorem

$$x\frac{du}{dx} + y\frac{du}{dy} = nu;$$

and when $y = x$, we have, in consequence of the symmetry,

$$2x\frac{du}{dx} - nu = 0 \ldots\ldots\ldots (1).$$

Again, if we suppose u to be expanded in series by powers of each of the variables

$$u = \Sigma . Q_i x^{n-i} . y^i = \Sigma . Q_i y^{n-i} . x^i,$$

from the symmetry. Substituting in the preceding equation, this gives

$$\Sigma . (2i - n) Q_i = 0 \ldots\ldots (2). \qquad \sigma.$$

2. To show that

$$\left(\frac{d}{dx}\right)^r (\epsilon^{ax} x^n) = \frac{x^{n-r}}{a^{n-r}} \left(\frac{d}{dx}\right)^n (\epsilon^{ax} x^r).$$

Expanding the differential on the left-hand side by the theorem of Leibnitz, we have

$$\left(\frac{d}{dx}\right)^r (\epsilon^{ax} x^n) = \left(d^r + rd^{r-1}d' + \frac{r(r-1)}{1.2} d^{r-2}d'^2 + \&c.\right) \left(\frac{\epsilon^{ax} x^n}{dx^r}\right),$$

where d refers to ϵ^{ax} and d' to x^n. Hence

$$\left(\frac{d}{dx}\right)^r (\epsilon^{ax} x^n)$$

$$= \left(a^r x^n + ra^{r-1} n x^{n-1} + \frac{r(r-1)}{1.2} a^{r-2} n(n-1) x^{n-2} + \&c\right) \epsilon^{ax},$$

$$= \frac{x^{n-r}}{a^{n-r}} \left(a^n x^r + na^{n-1} r x^{r-1} + \frac{n(n-1)}{1.2} a^{n-2} r(r-1) x^{r-2} + \&c.\right) \epsilon^{ax},$$

$$= \frac{x^{n-r}}{a^{n-r}} \left(d^n + nd^{n-1}d' + \frac{n(n-1)}{1.2} d^{n-2}d'^2 + \&c.\right) \frac{(\epsilon^{ax} x^r)}{dx^n},$$

$$= \frac{x^{n-r}}{a^{n-r}} \left(\frac{d}{dx}\right)^n (\epsilon^{ax} x^r),$$

by the theorem of Leibnitz. This theorem is true when one or both of the quantities n and r are fractional, so that we thus arrive at the curious result, that a differential to a fractional index may be expressed by means of a differential to an integer index of a function of the same form.

γ.

END OF VOL. II.

Fig. I

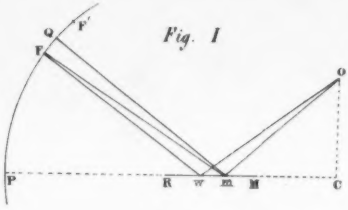
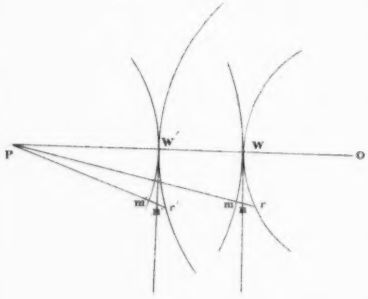


Fig. II



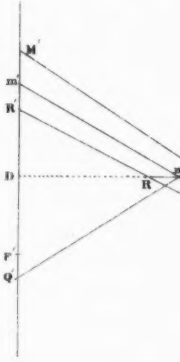
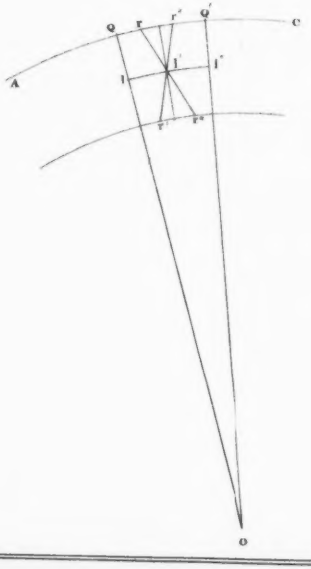
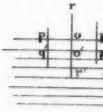


Fig. VI



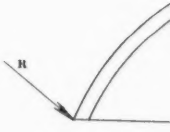


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Fig. IV

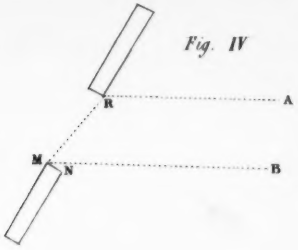

Fig. III

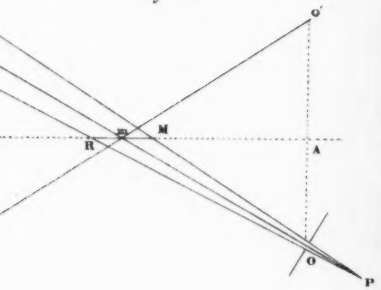

Fig. V

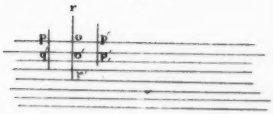
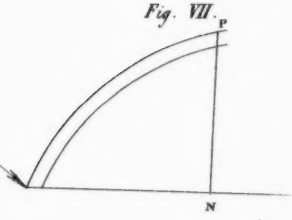
Fig. VII.